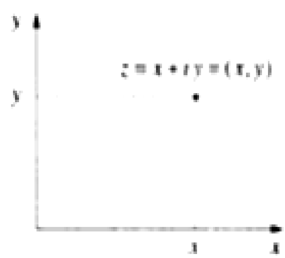
Definition of a Complex Variable

- Let D be the set of complex numbers. A rule f defined on D which assigns to every z in D, a complex number w, is called a function f or mapping f on D and it can be written as w = f(z). Here z is a complex variable and can be written as z = x + iy where x, y are real and $i = \sqrt{-1}$
- The set D is called domain of definition of f. The set of all w= f(z) (where z is an element of D) is called the range of f.
- If a and y are real variables, then z=a+iy is called a complex variable.

Representation of Argand plane

- The plane representing complex numbers as ordered pairs of real number is called complex plane and Argand plane or Gaussian plane.
- If corresponding to each value of a complex variable z=(x+iy) in a given region R,there correspond one or more values of another complex variable w(=u+iv),then w is called a function of the complex variable z and is denoted by,w = f(z) = u + iv
- Therefore w can be written as w = f(z) = u + iv where u and v are real



- Since $z = x + iy$ and z depends on x and y, u and v also depends on x and y. Hence, $u = u(x, y)$ and $v = v(x, y)$

- In general, $w = f(z) = u(x, y) + iv(x, y)$ where, u and v are real and imaginary parts of $w = f(z)$ respectively.

- If to each value of z,there corresponds one and only one value of w,then w is called a single-valued function of z.

- If to each value of z,there correspond more than one values of w, then w is called a multi-valued function of z.

Solved Problems

1) $w = f(z) = z^2$. Find the value at $z = 2 + i$

Solution :

Given $f(z) = z^2$

Let $z = x + iy$

$f(z) = (x + iy)^2 = x^2 - y^2 + i\,2xy$

Compare with $u + iv$

$u = x^2 - y^2$ real part

$v = 2xy$ imaginary part

$z = 2 + i$, comparing we get $x=2$, $y=1$

$u = 2^2 - 1^2 = 3$

$v = 2.2.1 = 4$

at $z = 2 + i$, $f(z) = 3 + 4i$

2) If $w = f(z) = z^2 + z$, find its real and imaginary parts. Also find f(z) at 1+i.

Solution :

$w = f(z) = z^2 + z = (x + iy)^2 + (x + iy) = (x^2 + y^2 + 2ixy) + (x + iy)$

$= (x^2 - y^2 + x) + i(2xy + y) = u + iv$ (say)

Then $u = x^2 - y^2 + x$ and $v = 2xy + y$ and $f(1+i) = (1+i)^2 + (1+i) = 1 + 3i$

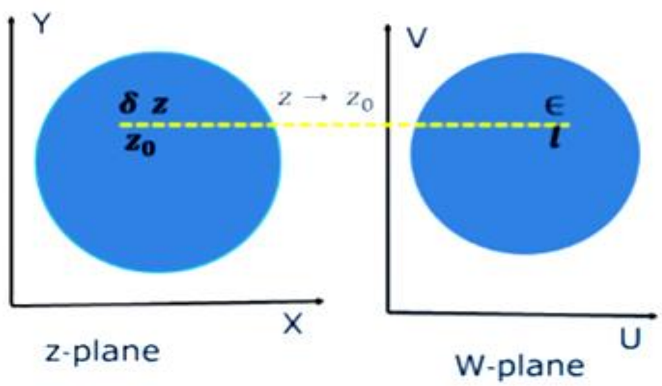
ϵ- disc around $w = w_0$

- Let w = u + iv represent all the complex numbers where u and v are real on a rectangular Cartesian coordinate plane (u, v). This is called the w-plane or (u, v) plane

- let w_0 be a point represented on this plane. Then, the set of all points w for which $|w - w_0| < \epsilon$ i.e., $\{w : |w - w_0| < \epsilon\}$ is called the ϵ- disc around w_0. This is also called as an ϵ- neighbourhood of w_0 .$\{w \,|\, 0 < |w - w_0| < \epsilon\}$ is called the deleted ϵ- disc around w_0

Limit of f(z)

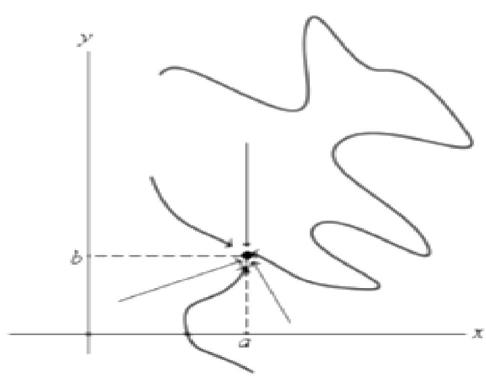
A function ω = f(z) is said to tend to limit l as z approaches a point z_0 , if for every real ϵ, we can find a positive δ such that $|f(z) - l| < \epsilon$ for $0 < |z - z_0| < \delta$
It is written as $\lim_{z \to z_0} f(z) = l$



Limit along five different paths :

- In order for the limit to exist, the function must be approaching the same value regardless of the path that we take as we move in towards (a, b)

- There are literally an infinite number of paths that we can take as we move in towards (a, b) . Here are a few examples of paths that we could take



- A couple of straight lines as well as a couple of stranger paths (which aren't straight line paths) are also shown
- Only 6 paths are included in the above graph
- By varying the slope of the straight line paths there are an infinite number of paths
- We will only consider the paths that aren't straight line paths
- In other words, to show that a limit exists we need to check an infinite number of paths and verify that the function is approaching the same value regardless of the path we are using to approach the point
- In graphical point of view, a function will be continuous at a point if the graph doesn't have any holes or breaks at that point
- This method is a very nice way to determine if the limit doesn't exist. If we can find two paths upon which the function approaches different values as we get near the point then we will know that the limit doesn't exist

Properties of Limits :

1. If limit of a function exist as $z \to z_0$, then it is unique
2. Let $f = u + iv$, $z = x + iy$, $z_0 = x_0 + iy_0$, then

   $$\lim_{z \to z_0} f(z) = u_0 + iv_0 \Leftrightarrow \lim_{\substack{x \to x_0 \\ y \to y_0}} = u_0 \text{ and } \lim_{\substack{x \to x_0 \\ y \to y_0}} = v_0$$

3. If $\lim_{z \to z_0} F(z) = f_0, \lim_{z \to z_0} G(z) = g_0$

   $$1)\ \lim_{z \to z_0} [F(z) \pm G(z)] = f_0 \pm g_0$$

   $$2)\ \lim_{z \to z_0} [F(z).G(z)] = f_0.g_0$$

   $$3)\ \lim_{z \to z_0} \left[\frac{F(z)}{G(z)}\right] = \frac{f_0}{g_0}, g_0 \neq 0$$

   $$4)\ \lim_{z \to z_0} c.F(z) = c.f_0$$

Solved Problems

1)Using the definition of limit, prove that $\lim_{z \to 1} \frac{z^2 - 1}{z - 1} = 2$

Solution :

let $f(z) = \dfrac{z^2 - 1}{z - 1}$

this function is not defined when $z = 1$

When $z \neq 1$, $f(z) = z + 1$. Thus $|f(z) - 2| = |z + 1 - 2| = |z - 1|$

$\therefore |f(z) - 2| < \epsilon$ whenever $|z - 1| < \epsilon$

Taking $\delta = \epsilon$, the condition for limit is satisfied for every $\epsilon > 0$.

$$\therefore \lim_{z \to 1} \frac{z^2 - 1}{z - 1} = 2$$

2)Show that $\lim_{z \to 2i}(2x + iy^2) = 4i$

Solution :

Consider $f(z) = 2x + iy^2$

$|f(z) - 4i| = |2x + iy^2 - 4i| \le |2x| + |y^2 - 4|$

$= 2|x| + |y-2||y+2|$

Let $\in > 0$ and $|x| < \in / 4, |y-2| < 1$

$\Rightarrow 2|x| < \frac{\in}{2}, |y+2| = |y-2+4| \le |y-2| + 4 < 5$

Consider $\min\left(\frac{\in}{10}, 1\right) = n$.

Then $|y-2| < n \Rightarrow |y-2||y+2| = \frac{\in}{10}.5 = \frac{\in}{2}$

We can find suitable δ for this $\in$.

$\therefore 0 < |z - 2i| < \delta \Rightarrow |2x + iy^2 - 4i| < \frac{\in}{2} + \frac{\in}{2} = \in$

$\therefore \lim_{z \to 2i}(2x + iy^2) = 4i$

5) Determine whether the limit exist or not. If they do exist, give the value of the limit.

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2}$$

Solution :

The given function is not continuous at origin.

Hence, we will use the path y = x. Along this path we have,

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x^3 x}{x^6 + x^2} = \lim_{(x,y)\to(0,0)} \frac{x^4}{x^6 + x^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^4 + 1} = 0$$

Now, lets try the path $y = x^3$. Along this path the limit becomes,

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x^3 x^3}{x^6 + (x^3)^2} = \lim_{(x,y)\to(0,0)} \frac{x^6}{2x^6} = \frac{1}{2}$$

We now have two paths that give different values for the limit and so the limit doesn't exist

6) Show that $\lim_{z\to 0} \frac{x^2 y}{x^4 + y^2}$ does not exist even though this function approaches the same limit along every straight line through the origin.

Solution :

Path I : $\lim_{\substack{x\to 0 \\ y\to 0}} \frac{x^2 y}{x^4 + y^2} = \lim_{y\to 0} 0 = 0$

Path II : $\lim_{\substack{y\to 0 \\ x\to 0}} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} 0 = 0$

Path III: Along any straight line through origin

Let $y = mx$

$$\lim_{\substack{y=mx \\ x\to 0}} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{mx^3}{x^4 + m^2 x^2} = \lim_{x\to 0} \frac{mx}{x^2 + m^2} = 0$$

Path IV: Let $y = mx^2$

$$\lim_{\substack{y=mx^2 \\ x\to 0}} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{mx^4}{x^4 + m^2 x^4} = \lim_{x\to 0} \frac{m}{1 + m^2} \neq 0$$ and different for different values of m.

Therefore the limit does not exist

7) Determine where the given function is continuous $(a)\frac{1}{1+z^2}\ (b)\frac{1}{z-1}$ inside a unit circle. How about in the complex plane.

Solution :

$\frac{1}{1+z^2}$ is continuous everywhere except where $1 + z^2 = 0 \Rightarrow z = \pm i$.

When unit circle is considered, $|z| < 1, z = \pm i$ are excluded. Thus $\frac{1}{1+z^2}$ is continuous inside $|z| = 1$

Similarly, $\frac{1}{z-1}$ is also continuous inside $|z| = 1$. If the entire complex plane is considered, both $\frac{1}{1+z^2}$ and $\frac{1}{z-1}$ are discontinuous, at $z = \pm i$ and $z = 1$ respectively.

8) Show that every differentiable function is continuous

Solution :

Let f(z) be differentiable at z_0. Then

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists}$$

$\therefore f(z_0)$ is well defined.

Consider, $\lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} . z - z_0$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} z - z_0 = f'(z_0) \lim_{z \to z_0} (z - z_0) = 0$$

Thus, $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} f(z_0) = f(z_0)$

$\therefore f(z)$ is continuous at z_0

Continuous Functions

- A function f(z) is said to be continuous at $z = z_0$ if $f(z_0)$ is defined and

$$\lim_{z \to z_0} f(z) = f(z_0)$$

- A function f is continuous at a point z_0, if corresponding to each positive number ϵ, a number $\delta > 0$ exist, such that

$$|f(z) - f(z_0)| < \epsilon \text{ whenever } |z - z_0| < \delta$$

Properties of continuous functions

1. If f is a continuous function of z at every point in a closed region R, then f is bounded in R
2. If F and G are continuous at z_0 then F+G, F-G, F.G, F/G $G \neq 0$ are also continuous
3. Any polynomial function is continuous
4. All trigonometric functions are continuous
5. If f(z) is continuous then $|f(z)|$ is also continuous

Solved Problems

1)Show that the function $f(z)=\frac{\bar{z}}{z}$ is not continuous at z = 0

Solution :

Let z = x + iy

Suppose $z \to 0$ along x-axis. Then we have y = 0, $z = x, \bar{z} = x$

$$\therefore \lim_{z\to 0}\frac{\bar{z}}{z} = \lim_{x\to 0}\frac{x}{x} = 1$$

Again suppose $z \to 0$ along y-axis.

Then $x = 0, z = iy$ and $\bar{z} = -iy$

$$\therefore \lim_{z\to 0}\frac{\bar{z}}{z} = \lim_{y\to 0}\frac{-iy}{iy} = -1$$

$\lim_{z\to 0}\frac{\bar{z}}{z}$ does not exist.

2)Show that the function $f(z) = \bar{z}$ is continuous over C.

Solution :

We have $|f(z)-f(z_0)| = |\bar{z}-\bar{z}_0|$. For any given $\in >0$ choose $\in = \delta$,we get

$|z-z_0| < \delta \Rightarrow |f(z)-f(z_0)| < \in$

$\therefore f(z)$ is continuous at z_0

Here z_0 is arbitrary. Thus $f(z)$ is continuous over C.

4) Show that f(z) = xy^2 + i(2x-y) is continuous for all z.

Solution:

Given f(z) = xy^2 + i(2x-y)

Consider f(z) = u(x, y) + iv(x, y) where u(x, y) = xy^2 and v(x, y) = 2x – y which are continuous everywhere.

Hence f(z) is continuous for all z.

5) Is the function defined by

$$f(z)=\begin{cases}\dfrac{z^2+3iz-2}{z+i} & for\ z\neq i\\ 5. & for\ z=-i\end{cases}$$ continuous. If not can the function be refined to make it continuous at $z=-i$?

Solution:

$f(z)=\dfrac{g(z)}{h(z)}$ is continuous when g(z) and h(z) are continuous except at h(z) = 0.

So f(z) is continuous everywhere except at z = -i, since g(z), h(z) are continuous.

Continuity at z = -i

$$\lim_{z\to -i} f(z)=\lim_{\substack{x\to 0\\ y\to -1}}\frac{(x+iy)^2+3i(x+iy)-2}{(x+iy)+i}=\lim_{y\to -1}\frac{-y^2-3y-2}{i(y+1)}=\lim_{y\to -1}\frac{-y-2}{i}=-\frac{1}{i}=i$$

Also, $\lim_{z\to -i} f(z)=\lim_{\substack{y\to -1\\ x\to 0}} f(z)=\lim_{x\to 0}\dfrac{2(x-i)+3i}{i}=i$

$\therefore \lim_{z\to -i} f(z)=i\neq 5=f(-i)$

$\therefore f(z)$ is not continuous at z = -i

Suppose we define f(z) as f(-i) = i instead of 5, then f(z) is continuous at z = -i and is therefore continuous everywhere.

This discontinuity at z = -i is known as removable discontinuity.

Derivative of f(z)

- Let $\omega = f(z)$, be a given function defined for all z in the neighbourhood of z_0.
- If $\lim_{z \to z_0} \dfrac{f(z_0+\Delta z)-f(z)}{\Delta z}$ exist, then the function f(z) is said to be derivable at z_0

Note:

- If a function is differentiable at a point, then it is continuous there
- A function can be continuous at a point, but not differentiable at that point

Solved Problems

1) Find the derivative of z^2

Solution :

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{2z.\Delta z + \Delta z^2}{\Delta z}$$
$$= \lim_{\Delta z \to 0} 2z + \Delta z = 2z$$

2) Find the derivative of $w = f(z) = z^3 - 2z$ at the point where (i) $z = z_0$ (ii) $z = 1$

Solution :

We have

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)^3 - 2(z_0+\Delta z) - (z_0^3 - 2z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0^3 + (\Delta z)^3 + 3z_0^2(\Delta z) + 3z_0(\Delta z)^2 - 2z_0 - 2\Delta z - z_0^3 + 2z_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[3z_0^2 + 3z_0\Delta z + (\Delta z)^2 - 2 \right] = 3z_0^2 - 2$$

In general, $f'(z) = 3z^2 - 2, \forall z$

(*ii*)Substituting $z_0 = 1$, we get $f'(1) = 3 - 2 = 1$

3) Show that the function $f(z) = z^n$, where n is a positive integer is differential for all values of z, where n is a positive integer

Solution :

Given $f(z) = z^n$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^n + nz^{n-1}\Delta z + \dfrac{n(n-1)}{2} z^{n-2}(\Delta z)^2 + \ldots + (\Delta z)^n - z^n}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[nz^{n-1} + \frac{n(n-1)}{2} z^{n-2}(\Delta z) + \ldots + (\Delta z)^{n-1} \right] = nz^{n-1}$$

Hence, $f'(z)$ exists for all values of z.

Properties of Differentiation

If f(z), g(z) are differentiable functions in a domain D, then

1) $\frac{d}{dz}\left[f(z) \pm g(z)\right] = \frac{d}{dz} f(z) \pm \frac{d}{dz} g(z)$

2) $\frac{d}{dz}\left[c.f(z)\right] = c.\frac{d}{dz} f(z)$

3) $\frac{d}{dz}\left[f(z).g(z)\right] = f(z)\frac{d}{dz} g(z) + g(z)\frac{d}{dz} f(z)$

4) $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)\frac{d}{dz} f(z) - f(z)\frac{d}{dz} g(z)}{\left[g(z)\right]^2}$

Definitions

Analytic function :

Let a function f(z) be derivable at every point z in the ϵ neighbourhood of z_0, then f(z) is said to be analytic at z_0

Entire function :

If f(z) is analytic at every point z on the complex plane, f(z) is said to be an entire function

Singular point :

A point at which an analytic function ceases to have derivative is called singular point. If $f'(z_0)$ does not exist at $z = z_0$ then $z = z_0$ is called a singular point of f(z)

Example :

If $f(z) = 1/z$ is analytic at every point $z \neq 0$

Then $f'(z) = -1/z^2$ if $z \neq 0$

At $z = 0$, $f'(z)$ does not exist

$z = 0$ is an isolated singular point of f(z).

Cauchy Riemann Equations

The necessary and sufficient conditions for the derivative of function f(z) = w = u(x, y) + i v(x, y) to exist for all values of $\bar{z}$ in the Region R are

1. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in R

2. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

The relations given by (ii) are known as Cauchy-Riemann equations or briefly C-R equations.

Proof :

Condition is necessary : Let (z) be analytic in the domain R.

We have to prove that u and v satisfy the equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ provided the partial derivatives exist

If f(z) possesses a unique derivative at point z then $f'(z) = \lim_{\Delta z \to 0} \frac{f(+\Delta z) - f(z)}{\Delta z}$ exists

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta y, y+\Delta y)] - [u(x, y) + iv(x, y)]}{\Delta x + i\Delta y} \to (1)$$

Since f′(z) exists and is same for all the directions along which Δz → 0.

Case – I:

Let Δz → 0 along X-axis, put Δz = Δx

Taking limit as Δz → 0 of (1), we get :

$$f'(z) = \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y+\Delta y) - u(x, y)}{\Delta x + i\Delta y} + i\frac{v(x+\Delta x, y+\Delta y) - v(x, y)}{\Delta x + i\Delta y} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x, y)}{\Delta x} \right]$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} [\because u_x + v_x \text{ exists as } f'(z) \text{ exists}]$$

Case –II :
Let $\Delta z = i\Delta y$

$$Now, f'(z) = \lim_{\Delta z \to 0}\left[\frac{u(x+\Delta x, y+\Delta y)-u(x,y)}{\Delta x + i\Delta y} + i\frac{v(x+\Delta x, y+\Delta y)-v(x,y)}{\Delta x + i\Delta y}\right]$$

$$\lim_{\Delta z \to 0}\left[\frac{u(x, y+\Delta y)-u(x,y)}{i\Delta y} + i\frac{v(x, y+\Delta y)-v(x,y)}{i\Delta y}\right]$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial v}{\partial y} \to (2)$$

Here f(z) cannot possibly be analytic unless these two limits are identical. Thus, a necessary condition that f(z) be analytic is $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$ *or*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial v}{\partial x} \ at\ (x, y) \to (3)$$

i.e. C-R equation must be satisfied

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Condition is Sufficient :

Suppose f(z) is a Single-valued function possessing partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of the region and C-R equation given by (3) are satisfied. By Taylor's theorem for a function of two variables, we have,

$$f(z+\Delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$$

$$= u(x,y) + \left[\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right] + \ldots + i\left[v(x,y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right) + \ldots\right]$$

$$= f(z) + \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]\delta y \ [\text{discarding terms of higher degree}]$$

$$\Rightarrow f(z+\delta z) - f(z) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]\delta y$$

Condition is Sufficient :

Now using C-R equation (3), in above we have,

$$f(z+\delta z) - f(z) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[\frac{-\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right]\delta y$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[\frac{i\partial v}{\partial x} + \frac{\partial u}{\partial x}\right]i\delta y$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right](\delta x + i\delta y) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta y$$

$$\therefore f'(z) = \lim_{\delta z \to 0}\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \text{ or } \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

which proves the sufficient conditions.

Maxima –Minima Principle

Theorem 1

If f is a continuous function defined on the closed interval [a,b],there is (at least) on point in [a,b] where f has a largest value, and there is one point where f has smallest value.

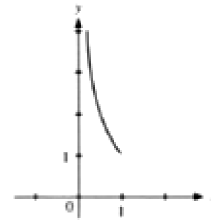
Consider a curve from the point corresponding to x=a to the point corresponding to x=b.

The curve has the highest point called the maximum value and the place has a low point called the minimum value.

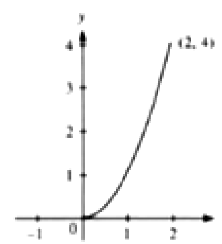
The theorem has two principal hypotheses.

1. The interval [a,b] is closed
2. The function f is continuous

- Consider the function f(x) =1/x is continuous in the open interval 0<x<1
- It has no maximum value in this open interval.



- Consider the function $f(x)= x^2$ is continuous in the open interval 0<x<2.
- This function has no maximum or minimum value in the open Interval.
- In closed Interval maximum value =4 and minimum value=0.



Definition

A function f(x) is said to have a maximum or minimum at x=a according as corresponding to a small positive number ∈ ,f(x)<f(a) or f(x)>f(a) for all x such that 0<|x-a|<∈

Working rule for finding the Maxima and Minima of a function

1. Find $f'(x)$ and equate it to zero. Let the roots of $f'(x)=0$ be $x=\alpha_r$, $r=1,2,\ldots..,n$.

2. Find $f''(\alpha_r)$. If $f''(\alpha_r)>0, x=\alpha_r$ is a minimum of $f(x)$. If $f''(\alpha_r)<0, x=\alpha_r$ is a maximum of $f(x)$.

3. If $f''(\alpha_r)=0$ for some r. Find $f'''(\alpha_r)\neq 0, x=\alpha_r$ is neither a maximum nor a minimum of $f(x)$. In this case $x=\alpha_r$ is called a point of inflection.

4 If $f'''(\alpha_r)\neq 0$ for some r. Find $f^{(iv)}(\alpha_r)$. If $f^{(iv)}(\alpha_r)>0$, $x=\alpha_r$ is a minimum of $f(x)$. If $f^{(iv)}(\alpha_r)<0$, $x=\alpha_r$ is a minimum of $f(x)$.

5 If $f^{(iv)}(\alpha_r)=0$ · Find $f^{(v)}(\alpha_r)$. If $f^{(v)}(\alpha_r)\neq 0$, $x=\alpha_r$ is neither a maximum nor a minimum of $f(x)$. If $f^{(v)}(\alpha_r)=0$, find $f^{(vi)}(\alpha_r)$. If $f^{(vi)}(\alpha_r)>0$, $x=\alpha_r$ is a minimum of $f(x)$ and if $f^{(vi)}(\alpha_r)<0$, $x=\alpha_r$ is a maximum of $f(x)$.

6 If $f^{(vi)}(\alpha_r)=0$, proceed to the next higher derivative as shown above in rule 5 and so on.

Problems:

1.Find the points of maxima and minima of the following functions.

$(i)2x^3 - 21x^2 + 36x + 10$

Solution:

$(i)2x^3 - 21x^2 + 36x + 10$

For maxima and minima we have,

$\frac{dy}{dx} = 0, i.e\ 6x^2 - 42x + 36 = 0$

$or \qquad x^2 - 7x + 6 = 0$

$or \qquad x^2 - 6x - x + 6 = 0$

$or \qquad x(x-6)-1(x-6)=0$

$or \qquad (x-1)(x-6)=0, \quad i.e$

$x=1,6$

$$\text{Now,} \qquad \frac{d^2y}{dx^2} = 12x - 42$$

Which is negative for x=1 and positive for x=6.Hence,x=1 is a point of maxima and x=6 is a point of minima.

$(ii) y = x^2 \log x$

For maxima and minima we have,

$\frac{dy}{dx} = 0, i.e\ 2x \log x + x^2 \frac{1}{x} = 0$

$or \qquad x(2\log x + 1) = 0$

$or \qquad 2\log x + 1 = 0 \quad as \quad x \neq 0$

$or \qquad \log x = -\frac{1}{2}$

$or \qquad x = e^{-\frac{1}{2}}$

Now,

$$\frac{d^2y}{dx^2} = 2\log x + 2x.\frac{1}{x} + 1 = 3 + 2\log x$$

$$At \quad x = e^{-\frac{1}{2}}, \frac{d^2y}{dx^2} = 3 + 2\log e^{-\frac{1}{2}} = 3 + 2\left(-\frac{1}{1}\right) = 2 > 0$$

Therefore=e-$^{1/2}$ is a point of minima.

2.Given $y=x^5+5x^3+5$,Find the extreme value of y,if any, where x is assumed to be a real variable.

Solution:

We have

$$y = x^5 + 5x^3 + 5$$

Then $$\frac{dy}{dx} = 5x^4 + 15x^2$$

Now, $$\frac{dy}{dx} = 0 \Rightarrow 5x^4 + 15x^2 = 0 \quad \text{or } x^2(x^2+3) = 0$$

Since x is a real variable,$x^2+3 \neq 0$

Hence x=0 is a possible point of extreme value.

We have

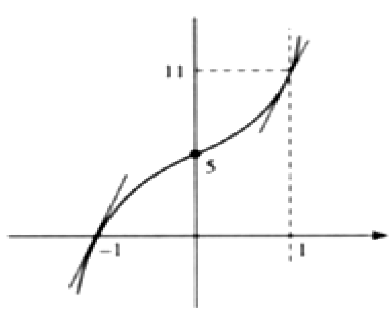
$$y'' = 20x^3 + 30x \quad and \quad y'(0) = 0$$

Hence x=0 is neither a point of maxima or minima.

We have

$$y''' = 60x^3 + 30 \quad and \quad y'''(0) = 30$$

Hence x=0 is a point of inflection.



Harmonic and Conjugate Harmonic Functions

Harmonic function :

- Solutions of Laplace equations are called harmonic functions (or) the functions which satisfy the laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

are known as Harmonic functions.

Laplacian operator :

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called Laplacian operator

Conjugate harmonic function :

- If two harmonic functions u and v satisfy the Cauchy Riemann equations in a domain D and they are the real and imaginary parts of an analytic function f(z) in D then u is said to be a conjugate harmonic function of u in D
- Two harmonic functions u and v which are such that u + iv is an analytic function are called conjugate harmonic functions
- If f(z) = u + iv is analytic and u and v satisfy laplace equation then u and v are called conjugate harmonic functions

Polar Form of a C-R equation

If $f(z) = f(re^{i\theta}) = u(r, \theta) + i\,v(r, \theta)$ and f(z) is derivable at $z_0 = r_0\, e^{i\theta}$ then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof :

Let $z = r e^{i\theta}$. Then, $f(z) = u(r, \theta) + iv(r, \theta)$

Differentiating it with respect to r partially, we get,

$$\frac{\partial}{\partial r} f(z) = f'(z)\frac{\partial z}{\partial r} = f'(z)e^{i\theta}$$

$$f'(z) = \frac{1}{e^{i\theta}}\frac{\partial f}{\partial r} = \frac{1}{e^{i\theta}}(u_r + iv_r) \rightarrow (1)$$

Similarly differntiating partially with respect to θ, we get,

$$\frac{\partial f}{\partial \theta} = f'(z)\frac{\partial z}{\partial \theta} = f'(z).r.ie^{i\theta}$$

$$\therefore f'(z) = \frac{1}{r.ie^{i\theta}}(u_\theta + iv_\theta) \rightarrow (2)$$

Using (1) and (2)

$$\therefore \frac{1}{e^{i\theta}}(u_\theta + iv_\theta) = \frac{1}{rie^{i\theta}}(u_\theta + iv_\theta)$$

$$\therefore u_r + iv_r = \frac{1}{r}\frac{\partial v}{\partial \theta} - i\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Hence, the result follows. These are called Cauchy-Riemann equation in polar form

Corollary :

If $f''(z)$ exists, then $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$ and $\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}=0$

Proof :

We have, $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}\rightarrow(1)$

and $\frac{\partial u}{\partial \theta}=\frac{\partial v}{\partial r}\rightarrow(2)$

Differentiating (1) partially with respect to 'r', we get,

$$\frac{\partial^2 u}{\partial r^2}=-\frac{1}{r^2}\frac{\partial v}{\partial \theta}+\frac{1}{r}\frac{\partial^2 v}{\partial r\partial \theta}$$

From (1), we have $\frac{1}{r}\frac{\partial u}{\partial r}=\frac{1}{r^2}\frac{\partial v}{\partial \theta}$

Differentiating (2) partially with respect to θ and multiplying with $\frac{1}{r}$, we get $\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=\frac{1}{r}\left(-\frac{\partial^2 v}{\partial r\partial \theta}\right)$

Adding the above three conditions, we get $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$

Similarly, we can prove that $\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}=0$

Note :

If $f'(z)$ exists (and also $|f''(z)|$)

(i) u, v satisfy Laplace's equations

(ii) u and v are harmonic functions

Solved Problems

1) (i) $f(z) = xy + iy$ is every where continuous but not analytic
(ii) $f(z_0) = x_0y_0 + iy_0$ is well defined for any $z_0 = x_0 + iy_0$

Solution :

(i) $\lim_{z\to z_0} f(z) = \lim_{z\to z_0} xy + iy$

$\lim_{\substack{x\to x_0\\ y\to y_0}} = x_0y_0 + iy_0 = f(z_0)$

$\lim_{\substack{y\to y_0\\ x\to x_0}} = x_0y_0 + iy_0 = f(z_0)$

$\therefore f$ is continuous everywhere

(ii) $f(z) = u + iv = xy + iy$
$u = xy,\ v = y$
$u_x = y,\ u_y = x,\ v_x = 0,\ v_y = 1$
C-R equations are not satisfied
f is not satisfied

2) (i)An analytic function with constant real part is constant

(ii)An analytic function with constant imaginary pat is constant

Solution :

(i) Let w = u + iv bean analytic function

By Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \to (1)$

By given data real part is constant.

Let $u = k_1$, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = 0$

From (1), we get $\frac{\partial u}{\partial y} = 0$ and $\frac{\partial v}{\partial x} = 0$

Thus, v is independent of x and y.

Therefore we can take $v = k_2$, a constant

$w = u + iv = k_1 + ik_2$ i.e. w = k , w is constant

(ii) similarly, we can show that an analytic function with constant imaginary part is constant.

3) Show that real and imaginary parts of an analytic function are harmonic

Solution :

Let f(z) = u + iv be an analytic function

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \to (1)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ by Cauchy-Riemann equation} \to (2)$$

Differentiating (1) partially with respect to x, $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \to (3)$

Differentiating (2) partially with respect to y, $\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} \to (4)$

(3) + (4)gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} = 0$

$\therefore u$ satisfy Laplace equation, hence u is harmonic

Again Differentiating (1) with respect to y, $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \to (5)$

Again Differentiating (2) with respect to x, $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial x^2} \to (6)$

(5) − (6) gives $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\therefore v$ satisfies Laplace equation. Hence, v is harmonic. Thus, both u and v are harmonic functions.

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \rightarrow (1)$

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ by Cauchy-Riemann equation $\rightarrow (2)$

Differentiating (1) partially with respect to x, $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x \partial y} \rightarrow (3)$

Differentiating (2) partially with respect to y, $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 v}{\partial y \partial x} \rightarrow (4)$

(3) + (4)gives, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 v}{\partial x^2} = 0$

$\therefore u$ satisfy Laplace equation, hence u is harmonic

Again Differentiating (1) with respect to y, $\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 v}{\partial y^2} \rightarrow (5)$

Again Differentiating (2) with respect to x, $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial x^2} \rightarrow (6)$

(5) − (6) gives $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

$\therefore v$ satisfies Laplace equation. Hence, v is harmonic. Thus, both u and v are harmonic functions.

5) Show that the function u(x, y) = e^x cosy is harmonic. Determine its harmonic conjugate v(x, y) and the analytic function f(z) = u + iv

Solution :

Given u(x, y) = e^x cos y

Differentiating with respect to x and y, we get

$\dfrac{\partial u}{\partial x} = e^x \cos y$ and $\dfrac{\partial u}{\partial y} = -e^x \sin y$

$\dfrac{\partial^2 u}{\partial x^2} = e^x \cos y$ and $\dfrac{\partial^2 u}{\partial y^2} = -e^x \cos y$

hence $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Thus, u is a harmonic function. Let v be the harmonic conjugate of u. Then by Cauchy-Riemann equations, we get,

$\dfrac{\partial u}{\partial x} = -\dfrac{\partial u}{\partial y} = e^x \sin y \rightarrow (1)$

Integrating $v = e^x \sin y + f(y)$

$\therefore \dfrac{\partial v}{\partial y} = e^x \cos y + f'(y) \rightarrow (2)$

Again $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = e^x \cos y$

From (2) and (3), we get,

$e^x \cos y = e^x \cos y + f'(y)$ or $f'(y) = 0 \Rightarrow f(y) = c$

Hence from (1), we get

$v = e^x \sin y + c$

$\therefore f(z) = u + iv = e^x \cos y + ie^x \sin y + ic = e^x(\cos y + i \sin y) + ic = e^x + ic = e^z + ic$

6) Find k such that $f(x, y) = x^3 + 3xky^2$ may be harmonic and find its conjugate

Solution :

We have $f(x, y) = x^3 + 3xky^2$

$$\therefore \frac{\partial f}{\partial x} = 3x^2 + 3ky^2, \frac{\partial f}{\partial y} = 6kxy \text{ and } \frac{\partial^2 f}{\partial x^2} = 6x, \frac{\partial^2 f}{\partial x^2} = 6kx.$$

Since f(x, y) is harmonic, therefore $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ i.e. $6x + 6kx = 0 \Rightarrow x(1+k) = 0 \Rightarrow x = 0, k = -1$

hence, $f(x) = x^3 - 3xy^2$

Let g(x, y) be the conjugate of $f(x, y)$. Then $dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = -\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy$ (using C-R equation)

$= -6kxy + (3x^2 + 3ky^2)dy$ (Q $k = -1$)

This is exact differential equation.

Integrating, $g = \int 6xy \, dx + \int -3y^2 \, dy + c$, y is constant

$$= 6y\left(\frac{x^2}{2}\right) - 3\left(\frac{y^3}{3}\right) + c = 3x^2y - y^3 - c$$

Milne-Thomson's Method

- By this method f(z) is directly constructed without finding v (or)u and the method is given below :

- Let f(z)= u(x, y) + iv(x, y)
- Since, z = x + iy, $\bar{z} = x - iy$ we have

$$x = \left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) \rightarrow (2)$$

Now considering this relation as a formal identity in two independent variables z and $\bar{z}$
Putting $\bar{z}$ = z in (2), we get

f(z)=u(z, 0) + iv (z, 0)- - - - (3)

Therefore (3) is same as (1), if we replace x by z and y by 0

Thus to express any function in terms of z, replace x by z and y by 0.

Now, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$, by Cauchy-Riemann equations

let $\frac{\partial u}{\partial x} = \phi_1(x,y)$ and $\frac{\partial u}{\partial x} = \phi_2(x,y)$. Then $f'(z) = \phi_1(x,y) - i\phi_2(x,y)$

Now, to express $f'(z)$ completely in terms of z, we replace x by z and y by 0 in (4)

$\therefore f'(z) = \phi_1(z,0) - i\phi_2(x,y)$

Hence, $f(z) = \int [\phi_1(z,0) - i\phi_1(z,0)]dz + c_1$

where, c_1 is a complex constant

Similarly if $v(x,y)$ is given, we can find u such that $u+iv$ is analytic. By using Milne-Thomsons method

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$, by C-R equations

$= \psi_1(x,y) + i\psi_2(x,y)$ where $\frac{\partial v}{\partial y} = \psi_1(x,y)$ and $\frac{\partial v}{\partial y} = \psi_2(x,y) = \psi_1(z,0) + i\psi_2(z,0)$

$\therefore f(z) = \int [\psi_1(z,0) - i\psi_2(z,0)]dz + c_2$

Solved Problems

1) Find the analytic function whose real part is e^{-x} (x sin y – y cos y)

Solution :

Let f(z) = u + iv, where u = e^{-x} (x sin y – y cos y)

Differentiating partially with respect to x and y, we get

$\frac{\partial u}{\partial x} = e^{-x}(\sin y) + (x\sin y - y\cos y)(-e^{-x})$ and $\frac{\partial u}{\partial y} = e^{-x}(x\cos y + y\sin y - \cos y)$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (using Cauchy-Riemann equation)

$= e^{-x}(\sin y - x\sin y + y\cos y) - ie^{-x}(x\cos y + y\sin y - \cos y)$

By Milne-Thomson's method, $f'(z)$ is expressed in terms of z by replacing x by z and y by 0

$\therefore f'(z) = 0 - e^{-z}(z-1) = -i(ze^{-z} - e^{-z})$

Integrating with respect to z, we have $f(z) = ize^{-z}$ + constant.

2) Find the analytic function whose imaginary part is e^x (x sin y + y cos y)

Solution :

Given $v = e^x$ (x sin y + y cos y) - - - - - - -(1)

Differentiating (1) partially with respect to x, $\frac{\partial v}{\partial x} = e^x\left[(x+1)\sin y + y\cos y\right]$

Differentiating (1) partially with respect to x, $\frac{\partial v}{\partial y} = e^x\left[(x+1)\sin y + y\cos y\right]$

But $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = e^x\left[(x+1)\cos y - y\sin y\right] + ie^x\left[(x+1)\sin y + y\cos y\right]$

Using Milne-Thomson method $f'(z) = e^z(z+1)$ [Putting $x = z$ and $y = 0$]

Integrating, $f(z) = ze^z + c$, i.e. $u + iv = (x+iy)e^{x+iy} + c = (x+iy).e^x.e^{iy} + c = e^x(x+iy)(\cos y + i\sin y) + c$

$\therefore u + iv = e^x(x\cos y - y\sin y) - ie^x(x\sin y + y + \cos y) + c$

Equating real parts, $u = e^x(x\cos y - \sin y) + c$

3) Find the conjugate harmonic function of the harmonic function $u = x^2 - y^2$

Solution :

Given $u = x^2 - y^2$ - - - - - -(1)

Differentiating (1) partially with respect to x, $\frac{\partial u}{\partial x} = 2x$

Again Differentiating, $\frac{\partial^2 u}{\partial x^2} = 2 \rightarrow (2)$

Differentiating (1) partially with respect to y, $\frac{\partial u}{\partial y} = -2y$

Again Differentiating, $\frac{\partial^2 u}{\partial y^2} = -2 \rightarrow (3)$

(2) + (3) gives $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore u$ is harmonic

let v be its harmonic conjugate $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 2x + i2y$

Using Milne-Thomson method, $f'(z) = 2z$

Integrating, we get $f(z) = z^2 + c$

$\therefore u + iv = (x + iy) + ik$ where $c = ik$

$f(z) = x^2 - y^2 + i2xy + ik = x^2 - y^2 + i(2xy + k)$

Equating imaginary parts $v = 2xy + k$ is the required form.

5) If f(z) = u + iv be an analytic function of z and if u − v = (x − y)(x² + 4xy + y²). Find f(z) in terms of z

Solution :

Given f(z) = u + iv - - - - - (1)

i.f(z)= iu − v - - - - - - - - -(2)

(1) + (2) gives, f(z) (i +1) = (u − v)+ i (u + v)

(1 + i) f(z)= U + iV where U = u − v, V = u + v

Differntiating $(1+i)f'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} + i\frac{\partial U}{\partial y} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} - i\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right)$

$= (x^2 + 4xy + y^2) + (x + y)(2x + 4y) - i\left[-(x^2 + 4xy + y^2) + (x - y)(4x + 2y)\right]$

$(1+i)f'(z) = 3x^2 + 6xy - 3y^2 - i\{3x^2 - 6xy - 3y^2\}$

Using Milne-Thomson method $(1+i)f'(z) = 3z^2 - i.3z^2$ (Putting $x = z$ and $y = 0$)

On Integrating, $(1+i)f(z) = (1-i)z^3 + c$

$f(z) = \frac{1-i}{1+i}z^3 + \frac{c}{1+i} = \frac{(1-i)^2}{1+1}z^3 + c = -iz^3 + c$

UNIT II : COMPLEX INTEGRATION

Finite integrals

Let us consider F(t) = U(t) + i V(t), $a \le t \le b$ - - - - - (1)
where U and V are real valued, sectionally continuous, functions of t in closed [a, b].
Each of the functions U(t)and V(t) is such that [a, b] can be divided into a finite number of sub intervals in each of which the functions are continuous and has finite limits from the interior at both end points of each interval

We define

$$\int_a^b F(t)\, dt = \int_a^b U(t)\, dt + i\int_a^b V(t)\, dt \text{ - - - - - - } (2)$$

Thus, $\int_a^b F(t)dt$ is a complex number such that

Real part of $\int_a^b F(t)dt = \int_a^b U(t)dt$

Imaginary part of $\int_a^b F(t)dt = \int_a^b V(t)dt$

Properties on finite integrals :

1. $\int_a^b F(t)dt = -\int_a^b F(t)dt$

2. $\int_a^b F(t)dt = \int_a^c F(t)dt + \int_a^b F(t)dt \quad a \le c \le b$

3. $\int_a^b k\, F(t)dt = k\int_a^b F(t)dt$

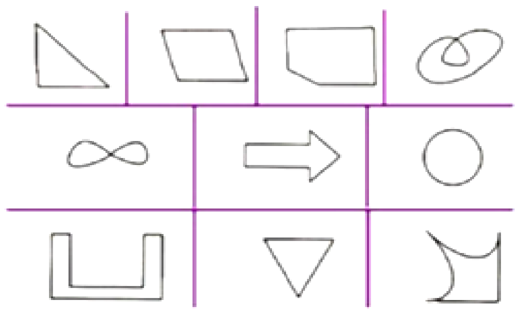
4. $\left|\int_a^b F(t)dt\right| \le \int_a^b \left|F(t)dt\right|$

Definition

- A set of points (x, y) such that x = x(t), y = y(t) $(a \le t \le b)$ where x(t), y(t) are continuous functions of the real variable t is called a continuous arc.

- If no two distinct values of t correspond to the same point (x, y), the arc is called a Jordan arc.

- If x(a) = x(b), y(a) = y(b) and if no other two values of t correspond to the same point (x, y), the continuous arc is a simple closed curve. A simple closed curve is also called a Jordan curve.
- Examples of simple closed curves

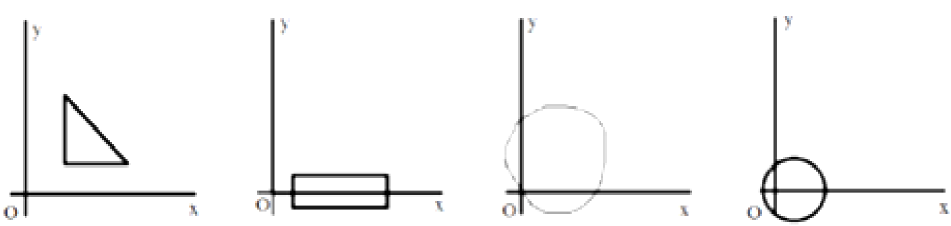


- If x(t), y(t) have continuous derivatives which do not vanish simultaneously for any value of t, then the arc has a continuously turning tangent. Then the arc is said to be smooth. Its length is given by the formula

$$L = \int_a^b \sqrt{[\dot{x}(t)]^2 + [\dot{y}(t)]^2}\, dt \text{ where } \dot{x}(t) = \frac{dx}{dt} \text{ and } \dot{y}(t) = \frac{dy}{dt}$$

- A contour is a continuous chain of a finite number of smooth arcs

- If a contour is closed and does not intersect itself, it is called a closed contour



Examples of closed contours

Line integral

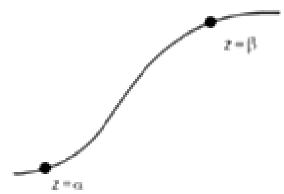
Let f(z) be a function of complex variable defined in a domain D

Let C be the arc in the domain joining from z = α to z= β

Let c be defined by x = x(t), y = y(t) (a ≤ t ≤ b) where α = x(a) + iy(a) and β = x(b) + iy(b).

Let x(t), y(t) be having continuous first order derivatives in [a, b]. Then line integral can be defined as

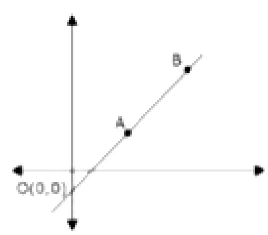
$$\int_c f(z)dz = \int_{t=a}^{b} f\left[x(t)+iy(t)\right]\left[\dot{x}(t)+i\dot{y}(t)\right]$$



Solved Problems

1) Integrate f(z) = x^2 + i xy from A(1, 1) to B(2, 8) along the straight line AB

Solution :



$$\int_c F(z)dz = \int_{(1,1)}^{(2,8)} (x^2+ixy)(dx+idy)$$

Equation of AB is $(y-1)(2-1)=(x-1)(8-1) \Rightarrow y = 7x-6 \Rightarrow dy = 7dx$

$$\int_{AB} F(z)dz = \int_{x=1}^{2} \left\{x^2+ix(7x-6)\right\}(dx+7idx) = \int_{x=1}^{2} \left\{x^2+i(7x^2-6x)\right\}(7i+1)dx$$

$$=(7i+1)\int_{x=1}^{2}\left[(7i+1)x^2-6ix\right]dx = (7i+1)\left[(7i+1)\frac{x^3}{3}-3ix^2\right]_1^2$$

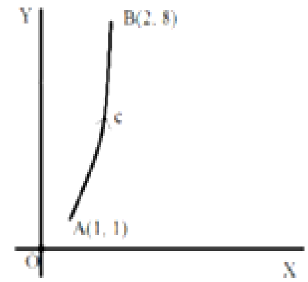
$$=(7i+1)\left[(7i+1)\frac{8}{3}-12i-\frac{(7i+1)}{3}+3i\right]=(7i+1)\left[(7i+1)\frac{7}{3}-9i\right]=(7i+1)(22i+7)$$

2) Integrate $f(z) = x^2 + i\,xy$ from the curves $c : x = t,\ y = t^3$ from $A(1, 1)$ to $B(2, 8)$

Solution :

Along c whose equations are $x = t,\ y = t^3$

$dx = dt,\ dy = 3t^2\, dt$



$A(1,1) \Rightarrow t = 1$

$B(2,8) \Rightarrow t = 2$

$$\int_c F(z)dz = \int_{(1,1)}^{(2,8)} (x^2 + ixy)(dx + idy)$$

$$\int_c F(z)dz = \int_{t=1}^{t=2} (t^2 + it^4)(dt + i3t^2 dt) = \int_1^2 (t^2 + it^4)(1 + i3t^2)dt = \int_1^2 (t^2 + it^4 + i3t^4 - 3t^6)dt = \int_1^2 (t^2 + (1+3i)t^4 - 3t^6)dt$$

$$= \left[\frac{t^3}{3} + (1+3i)\frac{t^5}{5} - \frac{3t^7}{7}\right]_1^2 = \frac{1}{45}(1441 + i549)$$

3) Evaluate $\int_{1+i}^{2+4i} z^2 dz$ along the parabola $x = t, y = t^2$ where $1 \le t \le 2$

Solution :

We have $\int_{1+i}^{2+4i} z^2 dz = \int_{(1,1)}^{(2,4)} (x + iy)^2 (dx + idy) = \int_{(1,1)}^{(2,4)} (x^2 - y^2 + i2xy)(dx + idy)$

$$= \int_{(1,1)}^{(2,4)} \left[(x^2 - y^2)dx - 2xydy\right] + i\int_{(1,1)}^{(2,4)} 2xydx + (x^2 - y^2)dy$$

along the parabola $x = t, y = t^2$ the points (1,1) and (2, 4) corresponds to t =1 and t = 2

$x = t,\ y = t^2$

$dx = dt,\ dy = 2tdt$

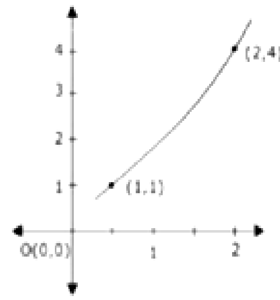
$$\int_{(1,1)}^{(2,4)} z^2 dz = \int_{t=1}^{2} (t^2 - t^4)dt - 2t.t^2.2tdt + i\int_{t=1}^{2} 2t.t^2 dt + (t^2 - t^4)(2tdt)$$

$$= \left(\frac{t^3}{3} - \frac{t^5}{5} - \frac{4t^5}{5}\right)_1^2 + i\left(\frac{2t^4}{4} + \frac{2t^4}{4} - \frac{2t^6}{6}\right)_1^2 = -\frac{86}{3} - 6i$$

4) Evaluate $\int_{1+i}^{2+4i} z^2 dz$ along the line joining 1 + i and 2 + 4i

Solution :

$$\int_{(1,1)}^{(2,4)} z^2 dz = \int_{(1,1)}^{(2,4)} \left[\left(x^2 - y^2\right) dx - 2xydy \right] + i \int_{(1,1)}^{(2,4)} 2xydx + (x^2 - y^2)dy$$



equation of the line joining (1, 1) and (2, 4)

$y = 3x - 2$

$dy = 3dx$

On substituting, we get

$$= \int_{(1,1)}^{(2,4)} (x^2 - (3x-2)^2)dx - 2x(3x-2).3dx + i \int_{(1,1)}^{(2,4)} \{(x^2 - (3x-2)^2)dx + x^2 - (3x-2)^2\} 3dx = \frac{86}{3} - 6i$$

5) Evaluate $\int z^2 dz$ along a straight line from $1 + i$ to $2 + i$ and then to $2 + 4i$

Solution :

$$\int_c F(z)dz = \int_c z^2 dz = \int (x^2 - y^2)dx - 2xydy + (x^2 - y^2)dy$$

From $1 + i$ to $2 + i$ i.e.(1,1) to (2,1)

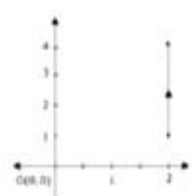
$y = 1$ and $dy = 0$

$$\int_c z^2 dz = \int_1^2 (x^2 - 1)dx + i\int_1^2 2xdx = \frac{4}{3} + 3i \rightarrow (1)$$

From $2 + i$ to $2 + 4i$ i.e. (2,1) to (2,4)

$x = 2$ and $dx = 04$

$$\int_c z^2 dx = \int_1^4 -4ydy + i\int (4 - y^2)dy = -30 - 9i \rightarrow (2)$$



(1) + (2) gives $\int_c z^2 dz = \left(\frac{4}{3} + 3i\right) + (-30 - 9i) = \frac{86}{3} - 6i$

8) Evaluate $\int_{(0,0)}^{(1,1)} (3x^2 - 4xy + ix^2)dz$ along $y = x^2$

Solution :

Let z = x + iy so that dz = dx + idy

$$\int_{(0,0)}^{(1,1)} (3x^2 + 4xy + ix^2)dz = \int_{(0,0)}^{(1,1)} (3x^2 + 4xy + ix^2)(dx + idy) ----(1)$$

Along $y = x^2, dy = 2xdx$

On putting the values of y and dy. (1) becomes

$$\int_{(0,0)}^{(1,1)} (3x^2 + 4xy + ix^2)dz = \int_0^1 (3x^2 + 4x^3 + ix^2)(dx + i2xdx) = \int_0^1 ((3+i)x^2 + 4x^3)(1 + i2x)dx$$

$$= \int_0^1 \left[(3+i)x^2 + 2(3+i)x^3 + i8x^4\right]dx = \left[(3+i)\frac{x^3}{3} + 2(3+i)\frac{x^4}{4} + i8\frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{3}(3+i) + \frac{2}{4}(3i+1) + i\frac{8}{5} = \frac{3}{2} + i\frac{103}{30}$$

9) Evaluate $\int_{(0,0)}^{(1,1)} (x^2 + y^2)dx - 2xydy$ along (i) $y = x$, (ii) $x = y^2$

Solution :

(i) Along the curve $y = x$ varies from 0 to 1 and $dy = dx$

$$\therefore \int_{(0,0)}^{(1,1)} (x^2 + y^2)dx - 2xydy = \int_0^1 (x^2 + x^2)dx - 2(x)(x)dx = \int_0^1 0dx = 0$$

(ii) Along the curve $y = x^2, dy = 2xdx$ and y varies from 0 to 1

$$\therefore \int_{(0,0)}^{(1,1)} (x^2 + y^2)dx - 2xydy = \int_0^1 (x^2 + x^4)dx - 2x(x^2)(2xdx) = \int_0^1 (x^2 - 3x^4)dx = \left(\frac{x^3}{3} - \frac{3x^5}{5}\right)_0^1 = \frac{1}{3} - \frac{3}{5} = \frac{5-9}{15} = \frac{-4}{15}$$

Cauchy Theorem

Let $f(z)=u(x,y)+iv(x,y)$ be analytic on and within a simple closed contour c and let $f'(z)$ be continuous there, then $\int_c f(z)dz=0$

Proof :

We have $f(z)=u(x,y)+iv(x,y)$ and $z=x+iy$

$dz=dx+iy$

$f(z)dz=(u+iv)(dx+idy)=(udx-vdy)+i(vdx+udy)$

$$\int_c f(z)dz=\int_c (udx-vdy)+i(vdx+udy)=\iint\left(-\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dxdy+i\iint\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)dxdy$$

By using Green's theorem in a plane

It is given that $f(z)=u+iv$ is analytic on and with in c

Hence $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

using C-R equations $\int_c f(z)dz=\iint_R 0dxdy+i\iint 0dxdy=0$

Hence the theorem

11) Show that $\int_{1-i}^{0-i}(3z+1)dz=0$

Solution :

$f(z)=3z+1$is analytic everywhere. Hence by Cauchy theorem $\int_c f(z)=0$

12) Show that $\oint_c \frac{dz}{z^2(z^2+16)}=0,\ 1\le|z|\le 2$

Solution :

$|z|=1$ is a circle with center (0,0) radius 1

$|z|=2$ is a circle with center (0,0) radius 2

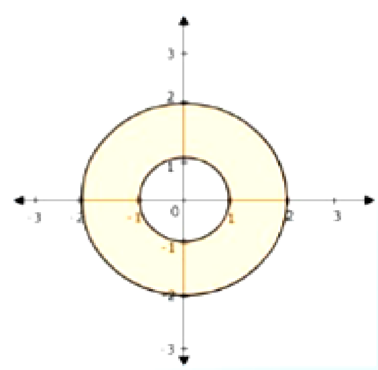
$1\le z\le 2$ is a portion which is in between the outer part of $|z|=1$ and inner part of $|z|=2$

$f(z)=\frac{1}{z^2(z^2+16)}$

Singular points of f(z) are $z=0$ and $z=\pm 4i$

These points are outside of the region $1<z<2$

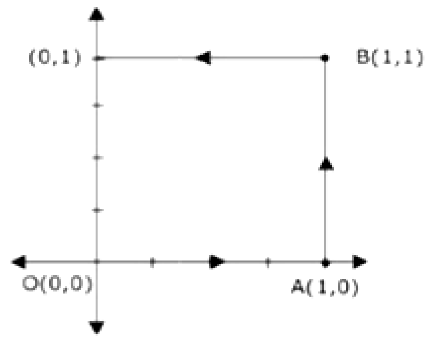
$\therefore f(z)$ is analytic within the region $\int_c f(z)dz=0$



13)Show that $\int_c (z+1)dz$ where c is the boundary of the square whose vertices at the points $z=0, z=1,$ $z=1+i, z=i$

Solution :

Let $I=\int_c (z+1)dz$ then $I = I_{OA}+I_{AB}+I_{BC}+I_{CD}$



Along OA (0,0) to (1,0)

x varies from 0 to 1

y varies from 0 to 0 $\Rightarrow y=0$ i.e. $z=x, dz=dx$

$$I_{OA}=\int_0^1 (x+1)dx = \left(\frac{x^2}{2}+x\right)_0^1 = \frac{1}{2}+1=\frac{3}{2}$$

Along AB (1,0) to (1,1)

x is from 1 to 1 $x=1, dx=0$

y is from 0 to 1

$$I_{AB}=\int_0^1 \left[(1+iy)+1\right] i dy = i\int_0^1 (2+iy)dy = i\left(2y+i.\frac{y^2}{2}\right)_0^1 = i\left(2+\frac{i}{2}\right)=2i-\frac{1}{2}$$

Along BC (1,1) to (0,1)

x varies from 1 to 0

y varies from 1 to 1, $y=1, dy=0$

$dz=dx$

$$I_{BC}=\int_0^1 (x+i+1)dx = \left(\frac{x^2}{2}+(i+1)x\right)_1^0 = 0-\left(\frac{1}{2}+i+1\right) = -\left(\frac{3}{2}+i\right)$$

Along CO (0,1) to (0,0)

x varies from 0 to 0, $x=0, dx=0$

y varies from 1 to 0

$$I_{CO}=\int_0^1 (iy+1)idy = i\int_0^1 (iy+1)dy = i\int_0^1 \left(i\frac{y^2}{2}+y\right)_1^0 = i\left[0-\frac{i}{2}-1\right]=\frac{1}{2}-i$$

Hence, $I = I_{OA}+I_{AB}+I_{BC}+I_{CO} = \frac{3}{2}+2i-\frac{1}{2}-\left(\frac{3}{2}+i\right)+\left(\frac{1}{2}-i\right)=0$

Cauchy Integral Formula

Let f(z) be an analytic function everywhere on and within a closed contour C. If z = a is any point within C then

$$f(a)=\frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)}dz$$

Generalization of Cauchy's Integral Formula :

If f(z) is analytic on and within a simple closed curve C and a is any point within C then

$$f^{n}(a) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{n-1}}\,dz$$

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)}\,dz$$

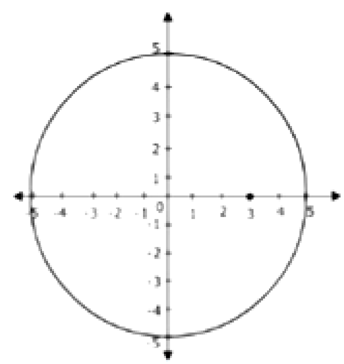
$$f'(a) = -\frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{2}}\,dz$$

$$f''(a) = \frac{2!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{3}}\,dz$$

$$f'''(a) = \frac{3!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{4}}\,dz$$

Solved Problems

14) Evaluate $\oint_c \frac{z^2+4}{z-3}\,dz$ where c is $|z| = 5$



Solution :

Comparing with $\int \frac{f(z)}{z-a}dz$

$z-3 = z-a \Rightarrow a = 3,\ f(z) = z^2+4$

$|z| = 5$ is a circle with center (0,0) and radius 5

$a = 3$ is within the circle

$$\int \frac{f(z)}{z-a}dz = 2\pi i f(a) = 2\pi i f(3) = 2\pi i[9+4] = 26\pi i$$

15) Evaluate $\oint_c \frac{z^2+4}{z-3}\,dz$ where $|z| = 2$

Solution :

$|z| = 2$ is a circle with center (0,0) and radius 2

The function $f(z) = \frac{z^2+4}{z-3}$ is analytic within c

By Cauchy's integral formula, $\int_c f(z)dz = 0$

16)Evaluate $\oint_c \frac{z^3+2z}{(z-a)^3}\,dz$

Solution :

Consider $\oint_c \frac{z^3+2z}{(z-a)^3}\,dz$

Let C is a closed contour

1)If a is within C then by Cauchy integral formula

$$f''(a) = \frac{\lfloor 2}{2\pi i}\int \frac{f(z)}{(z-a)^3}dz$$

$$f(z) = z^3 + 2z \Rightarrow f'(z) = 3z^2 + 2 \Rightarrow f''(z) = 6z$$

$$\oint_c \frac{f(z)}{(z-a)^3}\,dz = \pi i f''(a) = \pi i 6a$$

2)If a is outside the contour, then $f(z)$ is analytic $\Rightarrow \oint_c f(z)dz = 0$

17)Evaluate $\int_c \frac{e^{2z}}{(z-1)(z-2)}dz$ where c is a circle $|z| = 3$

Solution :

By partial fractions, $\frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$

$$\int_c \frac{e^{2z}}{(z-1)(z-2)}dz = \int_c \left(\frac{e^{2z}}{(z-2)} - \frac{e^{2z}}{(z-1)}\right)dz = \int_c \frac{e^{2z}}{(z-2)}dz - \int_c \frac{e^{2z}}{(z-1)}dz$$

$z = 1,2$ lies inside C, $\int_c \frac{f(z)}{z-a}dz = \left(2\pi i e^{2z}\right)_{z=1} = 2\pi i e^4 - 2\pi i e^2 = 2\pi i\left(e^4 - e^2\right)$

18)Evaluate $\int_c \frac{z}{z+1}dz$ where $\left|z + \frac{1}{z}\right| = 2$

Solution :

The function $\frac{z}{z^2+1} = \frac{z}{(z+i)(z-i)}$ has singular points at $z = \pm i$

The pole $z = i$ lies within $\left|z + \frac{1}{z}\right| = 2$

$z = -i$ lies within $\left|z + \frac{1}{z}\right| = 2$

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]dz = \frac{1}{2i}\int_c \frac{z}{z-i}dz - \frac{1}{2i}\int_c \frac{z}{z+i}dz = \frac{1}{2i}(2\pi i f(i)) - \frac{1}{2i}(2\pi i f(-i))$$

$$= \frac{1}{2i}2\pi i[i+i] = 2\pi i$$

19) Evaluate $\int_c \frac{dz}{z^8(z+4)}$ where c is a circle $|z| = 2$

Solution :

Let $f(z) = \frac{1}{z+4}$

$\int_c \frac{f(z)}{z^8} = \int_c \frac{f(z)}{(z-0)^8} dz \Rightarrow$ Here $a = 0$ is inside the circle $|z|$

Cauchy integral formula,

$$f^n(a) = \frac{\lfloor n}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n-1}} dz$$

Take $a = 0, n = 7$

$$\text{Hence, } f^n(0) = \frac{\lfloor n}{2\pi i}\int \frac{1}{z^8(z+4)} dz$$

$$\int_c \frac{dz}{z^8(z+4)} = \frac{2\pi i}{\lfloor n} f^7(0) = \frac{2\pi i}{\lfloor n}(-1)^7 \lfloor 7(z+4)^{-8} = \frac{-\pi i}{32768}$$

Power series

Definition of Power Series

- A series of the form $\Sigma\, a_n z^n$ is called a Power Series
- If $\Sigma\, a_n z^n$ converges at $z = z_1$, then it converges absolutely for all z such that $|z| < |z_1|$

Radius of convergence

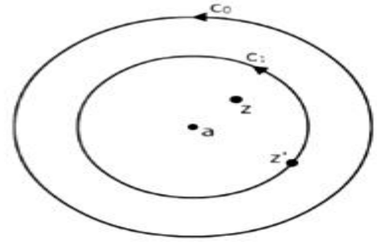
- If $\Sigma\, a_n z^n$ converges for $|z| < R$ and diverges for $|z| > R$, then R is called the radius of convergence of the power series and $|z| = R$ is called the circle of convergence of the power series
- If R is the radius of convergence of the power series $\Sigma\, a_n z^n$, then the power series converges uniformly for $|z| \leq R_1 < R$

Taylor Series

Let f(z) be analytic at all points within a circle C_0 with center at a and radius r. Then at each point z within C_0

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{\lfloor 2}(z-a)^2 + \ldots + \frac{f^n(a)}{\lfloor n}(z-a)^n + \ldots$$

The expansion of the right hand side of equation (1) is called the Taylor's series expansion of f(z) in powers of (z-a) or Taylor's series expansion of f(z) around z = a.



Taylor's Theorem

Solved Problems

1) Obtain the Taylor series expansion of f(z)=1/z about the point z=1 in |z -1| < 1

Solution :

$$Put\ z-1=\omega$$
$$\therefore z=1+\omega$$
$$f(z)=\frac{1}{z}=\frac{1}{1+\omega}=(1+\omega)^{-1}=1-\omega+\omega^2-\omega^3+.....=1-(z-1)+(z-1)^2-(z-1)^3+.....$$

2) Expand e^z as a Taylor series about z = 1

Solution :

$$\text{Put } z-1=\omega$$
$$\therefore z=1+\omega$$
$$e^z=e^{1+\omega}=e.e^{\omega}=e\left[1+\omega+\frac{\omega^2}{\lfloor 2}+\frac{\omega^3}{\lfloor 3}+.....\right]\left(\because e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+...\right)$$
$$=e\left[1+(z-1)+\frac{(z-1)^2}{\lfloor 2}+\frac{(z-1)^3}{\lfloor 3}+....\right]$$

3) Expand f(z)=1/z^2 in powers of z+1

Solution :

$$\text{Put } z+1=\omega \text{ and } z=\omega-1$$
$$f(z)=\frac{1}{z^2}=\frac{1}{(\omega-1)^2}=\frac{1}{(1-\omega^2)}=1+2\omega+3\omega^2+....+\ for\ |\omega|<1=1+2(z+1)+3(z+1)^2+.....$$

4) Obtain the Taylor's series to represent the function $\frac{z^2-1}{(z+2)(z+3)}$ in the region |z| < 2

Solution :

$$\text{Let } f(z)=\frac{z^2-1}{(z+2)(z+3)}=1+\frac{3}{z+2}-\frac{8}{z+3}=1+\frac{3}{2\left[1+\frac{3}{2}\right]}-\frac{8}{3\left[1+\frac{2}{3}\right]}=1+\frac{3}{2}\left[1+\frac{3}{2}\right]^{-1}-\frac{8}{3}\left[1+\frac{2}{3}\right]^{-1}$$
$$=1+\frac{3}{2}\left[1-\frac{3}{2}+\frac{3^2}{4}-\frac{3^3}{8}+...\right]-\frac{8}{3}\left[1-\frac{3}{3}+\frac{3^2}{9}-\frac{3^3}{27}+...\right]=1+\frac{3}{2}\left[\Sigma\frac{(-1)^n}{2^n}.z^n\right]-\frac{8}{3}\left[\Sigma\frac{(-1)^n}{3^n}.z^n\right]$$
$$=1+\Sigma(-1)^n\left[\frac{3}{2^{n+1}}-\frac{8}{3^{n+1}}\right].z^n$$

\

5) Expand log (1-z) when |z| < 1 using Taylor series

Solution :

Let $f(z) = \log(1-z), f(0) = 0$

$f'(z) = \frac{-1}{1-z}, f'(0) = -1$

$f''(z) = \frac{-1}{(1-z)^2}, f''(0) = -1$

$f'''(z) = \frac{-2}{(1-z)^3}, f'''(0) = -2$

By Taylor's theorem, about $z = 0$

$$f(z) = f(0) + zf'(0) + \frac{z^2}{\lfloor 2} f''(0) + \frac{z^3}{\lfloor 3} f'''(0) = 0 - z - \frac{z^2}{\lfloor 2} - \frac{2z^3}{\lfloor 3} = -\left[z + \frac{z^2}{2} + \frac{z^3}{3}\right]$$

6) Expand f(z)= sin z in Taylor series about z = ϖ/4

Solution :

By Taylor's theorem,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{\lfloor a} f''(a) + \dots + \frac{(z-a)^n}{\lfloor n} f^n(a) \rightarrow (1)$$

Put $a = \frac{\pi}{4}$ in (1), we get

$$f(z) = f\left(\frac{\pi}{4}\right) + \left(z - \frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \frac{\left(z - \frac{\pi}{4}\right)^2}{\lfloor 2} f''\left(\frac{\pi}{4}\right) + \dots + \dots \rightarrow (2)$$

$f(z) = \sin z, f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f'(z) = \cos z, f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f''(z) = -\sin z, f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$f'''(z) = -\cos z, f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

Substituting in (1),

$$f(z) = \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right) \cdot \frac{1}{\sqrt{2}} + \frac{\left(z - \frac{\pi}{4}\right)^2}{\lfloor 2} \cdot \left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(z - \frac{\pi}{4}\right)^3}{\lfloor 3} \cdot \left(-\frac{1}{\sqrt{2}}\right) + \dots$$

$$\sin z = \frac{1}{\sqrt{2}} \left[1 + \left(z - \frac{\pi}{4}\right) - \frac{\left(z - \frac{\pi}{4}\right)^2}{\lfloor 2} - \frac{\left(z - \frac{\pi}{4}\right)^3}{\lfloor 3} + \dots\right]$$

7) Expand $f(z)=\frac{z-1}{z-1}$ in Taylor's series about the point (i) z = 0, (ii) z = 1

Solution :

$$(i)\, f(z)=\frac{z-1}{z+1}=\frac{z+1-2}{z+1}=1-\frac{2}{z+1}=1-2.\frac{1}{z+1}$$

$$=1-2(1+z)^{-1}=1-2(1-z+z^2-z^3+....+...) \text{ if } |z|<1$$

$$=-1+2(z-z^2+z^3+....+...) \text{ if } |z|<1$$

$$=-1+2\sum_{n=1}^{\infty}(-1)^{n-1}z^n \text{ if } |z|<1$$

(ii) To expand $f(z)$ about $z=1$. Put $z-1=w \Rightarrow z=1+w$

$$\text{Hence, } f(z)=\frac{z-1}{z+1}=\frac{w}{1+w+1}=\frac{w}{2+w}=\frac{w}{2\left(1+\frac{w}{2}\right)}=\frac{w}{2}\left(1+\frac{w}{2}\right)^{-1}$$

$$=\frac{w}{2}\left[1-\frac{w}{2}+\left(\frac{w}{2}\right)^2-\left(\frac{w}{2}\right)^3+...\right] \text{ if } \left|\frac{w}{2}\right|<1=\frac{w}{2}-\left(\frac{w}{2}\right)^2+\left(\frac{w}{2}\right)^3-\left(\frac{w}{2}\right)^4+...+.. \text{ if } |w|<2$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{z-1}{2}\right) \text{ if } |z-1|<2$$

Laurent's Series Expansion

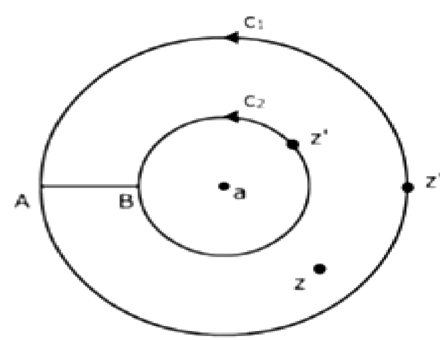
Let C_1 and C_2 be any two circles given by $|z' - a| = r_1$ and $|z' - a| = r_2$ respectively. Let z be any point in the ring shaped region between the circles C_1 and C_2 then,

$$f(z)=\sum a_n(z-a)^n+\sum\frac{b_n}{(z-a)^n}$$

where

$$a_n=\frac{1}{2\pi i}\int\frac{f(z')}{(z'-a)^{n+1}}dz'$$

$$b_n=\frac{1}{2\pi i}\int\frac{f(z)}{(z-a)^{-n+1}}dz'$$



Laurent's Theorem

Solved Problems

14) Expand $f(z) = \dfrac{1}{z^2 - 3z + 2}$ in the region $0 < |z-1| < 1$

Solution :

$$f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$z = 1, z = 2$ are singular points of $f(z)$

Put, $z - 1 = \omega, z = 1 + \omega$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{\omega - 1} - \frac{1}{\omega} = -\frac{1}{\omega} - \left[1 + \omega + \omega^2 + \omega^3 + ...\right] \text{ if } \omega < 1$$

$$= \frac{-1}{z-1} - \Sigma(z-1)^n \text{ if } 0 < |z-1| < 1$$

15) Find the Laurent series expansion of the function $f(z) = \dfrac{z^2 - 6z - 1}{(z-1)(z-3)(z+2)}$

Solution :

Using Partial fractions

$$\frac{z^2 - 6z - 1}{(z-1)(z-3)(z+2)} = \frac{1}{z-1} - \frac{1}{z-3} + \frac{1}{z+2} = \frac{1}{z+2-3} - \frac{1}{z+2-5} + \frac{1}{z+2}$$

$$= \frac{1}{(z+2)\left[1 - \frac{z}{z+2}\right]} + \frac{1}{5\left[1 - \frac{z+2}{5}\right]} + \frac{1}{z+2} = \frac{1}{z+2}.\left[1 - \frac{z}{z+2}\right]^{-1} + \frac{1}{5}\left[1 - \frac{z+2}{5}\right]^{-1} + \frac{1}{z+2}$$

$$= \frac{1}{z+2}\sum_0^{\infty}\left(\frac{z}{z+2}\right)^n + \frac{1}{5}\sum_0^{\infty}\left(\frac{z+2}{5}\right)^n + \frac{1}{z+2}$$

16) Find the Laurent series of $\dfrac{7z-2}{(z+1)z(z-2)}$ in $1<|z+1|<3$

Solution :

$$\text{Let } f(z)=\frac{7z-2}{(z+1)z(z-2)}$$

$$\text{Put } z+1=\omega, \text{ then } z=\omega-1 \Rightarrow f(z)=\frac{7(\omega-1)-2}{\omega(\omega-1)(\omega-1-2)}=\frac{7\omega-9}{\omega(\omega-1)(\omega-3)}$$

By Partial fractions,

$$f(z)=\frac{7\omega-9}{\omega(\omega-1)(\omega-3)}=-\frac{z}{\omega}+\frac{1}{\omega-1}+\frac{2}{\omega-3}=-\frac{z}{\omega}+\frac{1}{\omega\left(1-\frac{1}{\omega}\right)}-\frac{2}{3\left(1-\frac{\omega}{3}\right)}$$

$$=-\frac{z}{\omega}+\frac{1}{\omega}\left(1-\frac{1}{\omega}\right)^{-1}-\frac{2}{3}\left(1-\frac{\omega}{3}\right)^{-1}=-\frac{z}{\omega}+\frac{1}{\omega}\left(1+\frac{1}{\omega}+\frac{1}{\omega^2}+\dots\right)-\frac{2}{3}\left(1+\frac{\omega}{3}+\frac{\omega^2}{9}+\dots\right)$$

$$=-\frac{2}{z+1}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\dots-\frac{2}{3}\left[1+\frac{z+1}{3}+\frac{(z+1)^2}{3^2}\dots\right]$$

$\therefore$ The series is valid in the region $1<|z+1|<3$

17) Find the Laurent series expansion of $\dfrac{z^2-1}{(z+2)(z+3)}$ if $2<|z|<3$

Solution :

$$\text{Let } f(z)=\frac{z^2-1}{(z+2)(z+3)}=1-\frac{5z+7}{(z+2)(z+3)}=1+\frac{3}{z+2}-\frac{8}{z+3} \quad \text{(By partial fractions)}$$

$$\text{Given } 2<|z|<3 \Rightarrow \frac{2}{|z|}<1 \text{ and } \frac{|z|}{3}<1$$

$$f(z)=1+\frac{3}{z}\left[1+\frac{2}{z}\right]^{-1}-\frac{8}{3}\left[1+\frac{z}{3}\right]^{-1}=1+\frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{3}\right)^3+\dots\right]-\frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\dots\right]$$

$$=-\frac{5}{3}+\frac{3}{z}-\frac{6}{z^2}+\frac{12}{z^3}+\dots+\frac{8z}{9}-\frac{8z^2}{27}+\frac{8z^3}{81}+\dots$$

18) Find the Laurent series of the function $f(z)=\dfrac{z}{(z+1)(z+2)}$ **about z = -2.**

Solution :

$$f(z)=\frac{z}{(z+1)(z+2)}=z\left[\frac{1}{z+1}-\frac{1}{z+2}\right]$$

$$=\frac{z}{z+1}-\frac{z}{z+2}=\frac{(z+2)-2}{(z+2)-1}-\frac{(z+2)-2}{(z+2)}$$

$$=\frac{(z+2)-2}{-\left[1-(z+2)\right]}-1+\frac{2}{z+2}=\frac{2-(z+2)}{1-(z+2)}-1+\frac{2}{z+2}$$

$$=-1+\frac{2}{z+2}+\left[2-(z+2)\right]\left[1-(z+2)\right]^{-1}$$

$$=-1+\frac{2}{z+2}+\left[2-(z+2)\right]\left[1+(z+2)+(z+2)^2+\dots\right] \text{ if } |z+2|<1$$

19) Find the Laurent expansion of $\dfrac{1}{z^2-4z+3}$ **for (a) 1 < |z| < 3 (b) |z| < 1 (c) |z| > 3**

Solution :

Given $\dfrac{1}{z^2-4z+3}=\dfrac{1}{(z-1)(z-3)}=\dfrac{1}{2}\left[\dfrac{1}{z-3}-\dfrac{1}{z-1}\right]$.........(1)

(a)For $1<z<3$, we write (1) as

$$\frac{1}{z^2-4z+3}=\frac{1}{2}\frac{-1}{3\left(1-\frac{z}{3}\right)}+\frac{1}{2z\left(1-\frac{1}{z}\right)}=-\frac{1}{6}\left(1-\frac{z}{3}\right)^{-1}+\frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1}$$

$$=-\frac{1}{6}\left(1+\frac{z}{3}+\frac{z^2}{9}+\dots\right)+\frac{1}{2z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right)=-\frac{1}{6}\left(1+\frac{z}{3}+\frac{z^2}{9}+\dots\right)+\frac{1}{2}\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right)$$

which is a Laurent's series.

(b) If |z| < 1, then we write (1) as

$$\frac{1}{z^2-4z-3}=\frac{1}{2}\left[-\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1}+(1-z)^{-1}\right]$$

$$=\frac{1}{2}\left[-\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{9}+\dots\right)-(1+z^2+z^3+\dots)\right]$$

(c)If $|z|>3$, then $\left|\dfrac{3}{z}\right|<1$. So we write (1) as

$$\frac{1}{z^2-4z+3}=\frac{1}{2}\left[\frac{1}{z\left(1-\frac{3}{z}\right)}-\frac{1}{z\left(1-\frac{1}{z}\right)}\right]=\frac{1}{2z}\left[\left(1-\frac{3}{z}\right)^{-1}-\left(1-\frac{1}{z}\right)^{-1}\right]$$

$$=\frac{1}{2z}\left[\left(1+\frac{3}{z}+\frac{9}{z^2}+\dots\right)-\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right)\right]$$

20) Obtain the Laurent's series expansion of $f(z)=\frac{e^z}{z(1-3z)}$ about z = 1.

Solution :

To expand f(z) about z = 1, i.e., in powers of z -1, we put z − 1 = w.

Then

$$f(z)=\frac{e^z}{z(1-3z)}=\frac{e^{w-1}}{(1+w)(2+3w)}=-e^{w-1}\left[\frac{-1}{1+w}+\frac{3}{2+3w}\right]=e^{1-w}\left[\frac{1}{1+w}-\frac{3}{2+3w}\right]$$

$$=e.e^{w}\left[(1+w)^{-1}-\frac{3}{2}\left(1+\frac{3}{2}w\right)^{-1}\right]=e.e^{w}\left[\left(1-w+w^2-w^3+\ldots\right)-\frac{3}{2}\left(1-\frac{3}{2}w+\frac{9}{4}w^2-\ldots\right)\right]$$

$$=e^z\left[\left\{1-(z-1)+(z-1)^2-(z-1)^3+\ldots\right\}-\frac{3}{2}\left\{1-\frac{3}{2}(z-1)+\frac{9}{4}(z-1)^2-\frac{27}{8}(z-1)^3+\ldots\right\}\right]$$

Definitions

Singular points

A singular point (or singularity) of a function f(z) is the point at which the function f(z) ceases to be analytic

Singular points are of two types:

1. Regular singular point
2. Irregular singular point

1.Regular singular point :

Consider a second order homogeneous Linear Differential Equation with variable coefficient in canonical form.

$$y'' + P_1(x)y' + P_2(x)y = 0 \text{ - - - - - - } (1)$$

Let at least one of the functions P_1 and P_2 is not analytic at the singular point $x = x_0$ ($P_1 = \infty$ or $P_2 = \infty$) then redefine $P_1(x)$ and $P_2(x)$ as $Q_1(x) = (x - x_0)P_1(x)$ and $Q_2(x) = (x - x_0)^2 P_2(x)$. If $Q_1(x)$ and $Q_2(x)$ are analytic at $x = x_0$ then $x = x_0$ is said to be a regular singular point, otherwise an irregular singular point.

- A singular point $x = x_0$ is said to be a regular singular point if and only if $P_1(x)$ and $P_2(x)$ of the canonical form of second order homogeneous Linear Differential Equation with variable coefficient have removable discontinuity at $x = x_0$ and become analytic when the discontinuity is removed.

- In other words, after discontinuities are removed at $x = x_0$ the functions $P_1(x)$ and $P_2(x)$ have Taylor's series expansion about $x = x_0$.

2. Irregular singular point

- A singular point $x = x_0$ is said to be irregular singular point of homogeneous LDE of 2^{nd} order with variable coefficients if it is not a singular point.

Isolated singularity :

A point z = a is called an Isolated singularity of an analytic function f(z), if

(a) f(z) is not analytic at the point z = a

(b) f(z) is analytic in the deleted neighbourhood of z = a which contains no other singularity

Example :

If $f(z) = \dfrac{e^z}{z^2 + 4}$, then z = ± 2i are two isolated singular points of f(z).

If $f(z) = \dfrac{1}{\sin z}$, then z = ± π, ± 2π, ± 3π, ...are infinite number of isolated singular points of f(z).

Poles of an analytic function :

If z = a is an isolated singular point of an analytic function f(z), then f(z) can be expanded in Laurent's series about the point z = a, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

Poles of an analytic function :

- The series of negative integral powers of (z-a) namely, $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ is known as the principal part of the Laurent's series of f(z)
- If the principal part contains a finite number of terms, say m, (i.e. $b_n = 0$ for all n such that n > m), then the singular point z = a is called a pole of order m of f(z)

Example :

If $f(z) = \dfrac{z^2}{(z-1)(z+2)^2}$, then z = 1 is a simple pole and z = -2 is a pole of order 2.

Essential singularity:

- If the principal part of f(z) contains an infinite number of terms, i.e. the series $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ contains a infinite number of terms, then the point z = a is called essential singularity of f(z)

Example :

z = 0 is an essential singularity of $e^{1/z}$, since the principal part of $e^{1/z}$ contains infinite number of terms containing negative powers of (z – 0).

21) $x^2y''+axy'+by=0$

Solution:

Given $x^2y''+axy'+by=0 - - - -(1)$

The singular points are given by comparing (1) with , $A_0(x)y''+A_1(x)y'+A_2(x)y=0$

Here $A_0(x)=x^2, A_1(x)=ax, A_2(x)=b$

To get singular point, equate $A_0(x)=0, x^2=0 \Rightarrow x=0$

Expressing (1) in canonical form $y''+\frac{a}{x}y'+\frac{b}{x^2}y=0$

Here $P_1(x)=\frac{a}{x}$ and $P_2(x)=\frac{b}{x^2}$

$P_1(x)$ and $P_2(x)$ are not analytical at $x=0$

$Q_1(x)=(x-x_0)P_1(x)=(x-0)\times\frac{a}{x}=a, Q_2(x)=(x-x_0)^2P_2(x)=(x-0)^2\times\frac{b}{x^2}=b$

Clearly, $Q_1(x)$ and $Q_2(x)$ are both analytic at $x=0$

$\therefore x=0$ is a regular singular point.

At $x=2$:

$P_1(x)=\frac{1}{2}, P_2(x)=\infty$

$Q_1(x)=(x-x_0)P_1(x)=(x-2)\left(\frac{2}{x^2}\right)$

$Q_1(x)=0$ at $x=2$

$Q_2(x)=(x-x_0)^2P_2(x)=(x-2)^2\times\frac{x+3}{x^2(x-2)^2}$

$Q_2(x)$ at $x=2$ is $\frac{5}{4}$

$\therefore Q_1(x), Q_2(x)$ are analytic.

Therefore $x=2$ is a Regular Singular Point.

At $x = 0$:

$P_1(x) = \infty, P_2(x) = \infty$

$$Q_1(x) = (x-0)\,P_1(x) = x \times \frac{2}{x^2} = \frac{2}{x}$$

$$Q_2(x) = (x-0)^2 \times \frac{x+3}{x^2(x-2)^2} = \frac{x+3}{(x-2)^2}$$

At $x = 0, Q_1(x) = \infty$

$$Q_2(x) = \frac{3}{4}$$

$\therefore$ At $x = 0, Q_1(x), Q_2(x)$ are not analytic.

$\therefore x = 0$ is irregular singular point.

Residues

Definition:

The coefficient of $\frac{1}{z-a}$ in the expansion of f(z) about the isolated singularity z = a is called the residue of f(z) at that point. Thus from Laurent's series, the residue of $f(z) at\ z = a\ is\ b_1$. From Laurent series, the coefficient b_1 is given by

$$b_1 = \frac{1}{2\pi i}\int_C f(z)dz$$

$$i.e., \int_C f(z)dz = 2\pi i\ b_1 = 2\pi i \times [Residue\ of\ f(z) at\ z = a\ ] = 2\pi i\ [Res\, f(z)]_{z=a}$$

where C is a closed curve containing the point z=a (and such that f is analytic with in and on 'c')

Residue at a Pole:

If $f(z) has$ singularity (Pole) at the point $z = z_0\ and\ f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is its Laurent series about $z=z_0$ which is convergent in

$$0 < |z - z_0| < r.$$

Then $f(z) =$

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=-1}^{-\infty} a_n(z-z_0)^n$$

$$= \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$$

$$= \sum_{n=0}^{\infty} a_n(z-z_0)^n + [a_{-1}(z-z_0)^{-1} + a_{-2}(z-z_0)^{-2} + \ldots + a_{-m}(z-z_0)^{-3} + \ldots]$$

If the series with negative powers has a finite number of terms, say $m, (i.e,\ a_{-m} \neq 0, but\ a_{-m-1} = a_{-m-2} = a_{-m-3} = . \ . \ . = 0)$

then $z = z_0$ is called a pole of order m.

A pole of order one is called a simple pole. On the other hand, if the series has an infinite number of negative terms, then $z = z_0$ is called an essential singular point of $f(z)$

Consider the function $f(z) = e^{\frac{1}{z}}$

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} + \frac{1}{3!} \cdot \frac{1}{z^3} + . \ . \ .$$

This is an infinite series of negative powers of z-0 and therefore z=0 is an Essential Singularity.

Cauchy's Residue Theorem

Statement:

If $f(z)$ is analytic with in and on a closed curve C, except at a finite number of poles $z_1, z_2, z_3 . \ . \ . \ z_n$ within C and $R_1, R_2, . \ . \ . \ R_n$ be the residues of $f(z)$ at these poles then

$$\int_C f(z)dz = 2\pi i\ (R_1 + R_2 + . \ . \ . + R_n)\ or$$

$$\int_C f(z)dz = 2\pi i \times \text{sum of the residues at the poles with in c}$$

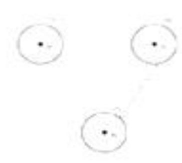
Proof:

Suppose $r_1, r_2, . \ . \ . \ r_n$ *be the* circle with center at $z_1, z_2 . \ . \ . \ z_n$ respectively and their radii so small so that they lie entirely within closed curve c and do not overlap.

Let $f(z)$ is analytic within the region enclosed by the curve C between these circles.

∴By Cauchy's theorem for multiple connected regions we have

$$\int_C f(z)dz = \int_{r_1} f(z)dz + \int_{r_2} f(z)dz + . \ . \ . + \int_{r_n} f(z)dz$$



Residue at a pole of order m

Statement: If f(z) is analytic with in a curve C and has a pole of order **m** at $z = z_0$ then the residue at $z = z_0$ is

$$\underset{z\to z_0}{\text{Lt}} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)]$$

Proof:

Given that $f(z)$ has a pole of order m.

Therefore $f(z)$ is expressible as $(z-z_0)^m f(z) = \emptyset(z)$

Where $\emptyset(z)$ is analytic and $\emptyset(z_0) \neq 0$

$$\therefore\ f(z) = \frac{\emptyset(z)}{(z-z_0)^m} \qquad \rightarrow (1)$$

Residues of $f(z) at\ z = z_0\ is\ a_{-1}$ where

$$a_{-1} = \frac{1}{2\pi i}\int_C f(z)dz = \frac{1}{2\pi i}\int_C \frac{\emptyset(z)}{(z-z_0)^m}dz$$

$$= \frac{1}{(m-1)!}\emptyset^{m-1}(z_0)\left[Since\ f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right]$$

$$\therefore a_{-1} = Res\,(f: z = z_0)\ or\ [Res\ f(z)]_{z=z_0}\ \underset{z\to z_0}{\text{Lt}}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)][by(1)]$$

Residue at Infinity:-

If $f(z)$ has an isolated singularity at $z = \infty$ or is analytic there then the residue at $z = \infty$ is defined as

$$\text{Res } (f: z = \infty)\ or\ [Res.\, f(z)]_{z=\infty} = -\frac{1}{2\pi i}\int_C f(z)dz$$

Note:

1. The residue of $f(z)$ at $z = \infty$ is the negative of the coefficient of $\frac{1}{z}$ in the expansion of $f(z)$ for values of z in the neighbourhood of $z = \infty$.
2. The residue of f(z) at $z = \infty$ is $\underset{z\to\infty}{\text{Lt}}\{-zf(z)\}$ provided f(z) is analytic at $z = \infty$

Solved Problems

1) Determine the poles of the function $(i).\ \frac{z}{\cos z}$ $(ii).\ \cot z$

Solution:

(i). The poles of $f(z) = \frac{z}{\cos z}$ are given by equating denominator to zero

$i.e., \cos z = 0$

$i.e.\ z = (2n+1)\frac{\pi}{2}$, n' being '0' or an integer

$$i.e.\ z = (2n+1)\frac{\pi}{2}, n = 0, \pm 1, \pm 2$$

Hence these are simple poles of $f(z)$

(ii). The poles of $f(z) = \cot z = \frac{\cos z}{\sin z}$ are given by equating denominator to zero

. $i.e., \sin z = 0$

$$i.e.\ z = n\pi, n = 0, \pm\pi, \pm 2\pi \ . \ . \ .$$

Which are simple poles of $f(z)$

2) Determine the poles of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$

and the residues at each pole

Solution:

The poles of $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ are given by equating denominator to zero i.e., $z = 1$ and $z = -2$ are the zeros of denominator of order 2 and 1

$\therefore z = 1$ is a pole of order '2' and $z = -2$ is a pole of order 1 of $f(z)$

$$[Res\ f(z)]_{z=1} = \underset{z\to 1}{\text{Lt}} \frac{1}{1!}\frac{d}{dz}\ [(z-1)^2 f(z)]$$

$$= \underset{z\to 1}{\text{Lt}} \frac{d}{dz}\left[(z-1)^2 \frac{z^2}{(z-1)^2(z+2)}\right] = \underset{z\to 1}{\text{Lt}}\left[\frac{d}{dz}\left(\frac{z^2}{z+2}\right)\right]$$

$$= \underset{z\to 1}{\text{Lt}}\left[\frac{(z+2).2z - z^2.1}{(z+2)^2}\right] = \underset{z\to 1}{\text{Lt}}\left[\frac{z^2+4z}{(z+2)^2}\right] = \frac{5}{9}$$

4) Find the residue at $z = 0$ of the function $f(z) = \frac{1+e^z}{\sin z + z\cos z}$

Solution:

The residue of $f(z)$ at $z = 0$ is

$$\underset{z\to 0}{\text{Lt}}\{z\, f(z)\} = \underset{z\to 0}{\text{Lt}}\ z.\frac{1+e^z}{z\left(\cos z + \frac{\sin z}{z}\right)} = \underset{z\to 0}{\text{Lt}} \frac{1+e^z}{\cos z + \frac{\sin z}{z}}$$

$$= \frac{2}{2} = 1 \left(\because \underset{z\to 0}{\text{Lt}} \frac{\sin z}{z} = 1\right)$$

5) Find the residues of the function $f(z) = \frac{1-e^{2z}}{z^4}$ at the poles

Solution:

Given $f(z) = \frac{1-e^{2z}}{z^4}$

Here $z = 0$ Is the singular point of $f(z)$

Expanding $f(z) = \frac{1-e^{2z}}{z^4} = \frac{1-\left[1+\frac{2z}{1!}+\frac{4z^2}{2!}+\frac{8z^3}{3!}+\ldots\right]}{z^4}$

$= -\left[\frac{2}{z^3}+\frac{2}{z^2}+\frac{4}{3}\cdot\frac{1}{z}+\frac{2}{3}+\frac{4}{15}z+\ldots\right]$

$z = 0$ is a pole of order 3 ,because $\frac{1}{z^3}$ is the highest negative power of $(z-0)$

Therefore, the residue of $f(z)$ at $z = 0$ is $\frac{-4}{3}$

7) Find the residue of $\frac{z^2}{z^4+1}$ at these singular points which lie inside the circle $|z| = 2$

Solution:

Let $f(z) = \frac{z^2}{z^4+1}$

Poles of $f(z)$ are obtained by equating the denominator to zero i.e$z^4+1=0$

$or\ z^4 = -1\ or\ z = (-1)^{\frac{1}{4}} = (Cos\,\pi + i\sin\pi)^{\frac{1}{4}} = \cos\left(\frac{2n\pi+\pi}{4}\right) + i\sin\left(\frac{2n\pi+\pi}{4}\right)$ Where n=0, 1, 2, 3

∴The four values of z are

$Cos\frac{\pi}{4} + i\sin\frac{\pi}{4}, Cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}, \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\ and\ \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$

$i.e.\ \frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}, \frac{-1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$

Hence the simple poles of $f(z)$ are $\frac{\pm 1 \pm i}{\sqrt{2}}$

And all these lies within the circle $|z| = 2$ with center 0 and radius 2.

Now let

$$f(z) = \frac{z^2}{z^4+1} = \frac{\phi(z)}{\Psi(z)}$$

$$\therefore\ [Res\ f(z)]_{z=\frac{1+i}{\sqrt{2}}} = \frac{\phi\left(\frac{1+i}{\sqrt{2}}\right)}{\Psi'\left(\frac{1+i}{\sqrt{2}}\right)} = \frac{i}{4i\left(\frac{1+i}{\sqrt{2}}\right)} = \frac{1}{2\sqrt{2}(1+i)} = \frac{1-i}{4\sqrt{2}}\ \left[\because\ [Res\ f(z)]_{z=z_0} = \frac{\phi(z_0)}{\Psi(z_0)}\right]$$

Here $\Psi(z) = z^4+1$ and $\Psi'(z) = 4z^3$

$$[Res\ f(z)]_{z=\frac{1+i}{\sqrt{2}}} = \frac{\phi\left(\frac{-1+i}{\sqrt{2}}\right)}{\Psi\left(\frac{-1+i}{\sqrt{2}}\right)} = \frac{-i}{4(-i)\left(\frac{-1+i}{\sqrt{2}}\right)} = \frac{1}{2\sqrt{2}(-1+i)} = \frac{(1+i)}{4\sqrt{2}}$$

$$[Res\ f(z)]_{z=\frac{1+i}{\sqrt{2}}} = \frac{\phi\left(\frac{-1-i}{\sqrt{2}}\right)}{\psi'\left(\frac{1-i}{\sqrt{2}}\right)} = \frac{i}{4i\left(\frac{-1-i}{\sqrt{2}}\right)} = \frac{-1}{2\sqrt{2}(1+i)} = \frac{-(1-i)}{4\sqrt{2}}$$

$$[Res\ f(z)]_{z=\frac{1+i}{\sqrt{2}}} = \frac{\phi\left(\frac{1-i}{\sqrt{2}}\right)}{\psi'\left(\frac{1-i}{\sqrt{2}}\right)} = \frac{-i}{-4i\left(\frac{1-i}{\sqrt{2}}\right)} = \frac{1}{2\sqrt{2}(1-i)} = \frac{1+i}{4\sqrt{2}}$$

8) Find the residue of $(i). \frac{z^2-2z}{(z+1)^2(z^2+1)}$ $(ii). \tan z$ at each pole

Solution:

Let $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+1)} = \frac{z^2-2z}{(z+1)^2\,(z+i)(z-i)}$

$\therefore$ Poles of $f(z)$ are -1,i,-i

Observe that -1 is a pole of order 2 and poles $\pm i$ are of order one

$$\therefore [Res\ f(z)]_{z=-1} = \frac{1}{(2-1)!} \lim_{z\to -1}\left[\frac{d}{dz}(z+1)^2 f(z)\right] = \lim_{z\to -1}\left[\frac{d}{dz}\left(\frac{z^2-2z}{z^2+1}\right)\right]$$

$$= \lim_{z\to -1}\left[\frac{(z^2+1)(2z-2)-(z^2-2z)(2z)}{(z^2+1)^2}\right] = \frac{-1}{2}$$

$$[Res\ f(z)]_{z=i} = \lim_{z\to i}[(z-i)f(z)] = \lim_{z\to i}\left[\frac{z^2-2z}{(z+1)^2(z+i)}\right] = \frac{i^2-2i}{(i+1)^2(2i)} = \frac{1+2i}{4}$$

$$[Res\ f(z)]_{z=-i} = \lim_{z\to -i}[z+i]f(z) = \lim_{z\to -i}\left[\frac{z^2-2z}{(z+1)^2(z-i)}\right] = \frac{(-i)^2+2i}{(-i+1)^2(-2i)} = \frac{2i-1}{(-2i)^2} = \frac{2i-1}{(-2i)^2}$$

$$= \frac{2i-1}{-4} = \frac{1-2i}{4}$$

Let $f(z) = \tan z = \frac{\sin z}{\cos z}$

$\therefore$ poles of $f(z)$ are given by $\cos z = 0$

$i.e., z = (2n+1)\frac{\pi}{2}, \quad where\ n = 0, \pm 1, \pm 2, \pm 3 \ldots$

All these poles are simple poles of $f(z)$ and denoting each pole by 'a' we have

$$[Res\ f(z)]_{z=a} = \mathop{Lt}_{z\to a}(z-a)f(z) = \mathop{Lt}_{z\to a}\frac{(z-a)\sin z}{\cos z}\left(=\frac{0}{0}\right) = \mathop{Lt}_{z\to a}\frac{(z-a)\cos z+\sin z}{-\sin z} = -1$$

(Using L Hospital's rule and then putting z=a)

Hence residue of $f(z)$ at each of the poles is -1

Evaluation of integrals using Residue Theorem

Step 1:

Consider the given region

Step 2:

Calculate the pole of the function

Step 3:

Consider the poles which are within the region

Step 4:

Evaluate the residue at each pole within the region

Step 5:

Use the formula $\int_C f(z)dz = 2\pi i\,(R_1 + R_2 + . \;.\; . + R_n)$

Solved Problems

1) Evaluate

$\oint_c \frac{2z-1}{z(z+2)(2z+1)}\,dz$ where c is the circle $|z| = 1$

Solution:

Here

$$f(z) = \frac{2z-1}{z(z+2)(2z+1)}$$

has 3 simple poles at $z = 0, z = -2$ and $z = -\frac{1}{2}$

But the only poles $z = 0$ and $z = -\frac{1}{2}$

Lies inside the circle $|z| = 1$ Since $|z| - 1 < 0$ for $z = -\frac{1}{2}$

$$\therefore\ [Res\ f(z)]_{z=0} = \underset{z\to 0}{\text{Lt}}\, z\, f(z) = \underset{z\to 0}{\text{Lt}} \frac{2z-1}{(z+2)(2z+1)} = -\frac{1}{2.1} = -\frac{1}{2}$$

$$[Res\ f(z)]_{z=-\frac{1}{2}} = \underset{z\to -\frac{1}{2}}{\text{Lt}}(2z+1)f(z) = \underset{z\to -\frac{1}{2}}{\text{Lt}} \frac{2z-1}{z(z+2)} = -\frac{2}{-\frac{3}{4}} = \frac{8}{3}$$

∴ By residue theorem we have

$$\oint_c \frac{2z-1}{z(z+2)(2z+1)} = 2\pi i\left(-\frac{1}{2}+\frac{8}{3}\right) = \frac{13}{3}\,\pi i$$

2) Evaluate $\oint_c \tan z\, dz$ where c is the circle $|z| = 2$.

Solution: Given $f(z) = \tan z = \frac{\sin z}{\cos z}$

The poles of $f(z)$ are given by $\cos z = 0$

$i.e.\, z = \pm (2n+1)\frac{\pi}{2}, \quad n = 0,1,2,....$

Out of these only $z = \pm\frac{\pi}{2}$ (± 1.570) lies inside $c: |z| = 2$

$$\therefore \operatorname{Res}_{z=\pm\frac{\pi}{2}} f(z) = \operatorname*{Lt}_{z\to\pm\frac{\pi}{2}} \left(z \pm \frac{\pi}{2}\right) . \frac{\sin z}{\cos z} = \operatorname*{Lt}_{z\to\pm\frac{\pi}{2}} \frac{\sin z + \left(z \pm \frac{\pi}{2}\right).\cos z}{-\sin z} = -1$$

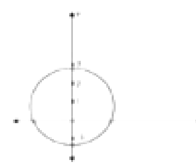
(Using L Hospital's rule)

By residue theorem $\oint_c \tan z\, dz = 2\pi i\,[-1-1] = -4\pi i$

3) Evaluate $\oint_c \frac{dz}{(z^2+4)^2}$ where $c: |z - i| = 2$

Solution: The integrand $\frac{1}{(z^2+4)^2}$ has double poles $z = \pm 2i$ of these poles only $z = 2i$ lies inside 'C'

Again



$$\oint_c \frac{dz}{(z^2+4)^2} = \oint_c \frac{1}{(z+2i)^2} . \frac{d}{(z-2i)^2}$$

Since $\frac{1}{(z+2i)^2}$ is analytic in c applying Cauchy's integral formula for derivatives we have

$$\therefore \oint_c \frac{dz}{(z^2+4)^2} = \frac{2\pi i}{1!} \frac{d}{dz}\left[\frac{1}{(z+2i)^2}\right]_{z=2i} = 2\pi i \left[\frac{-2}{(z+2i)^3}\right]_{z=2i} = \frac{-4\pi i}{4^3 i^3} = \frac{\pi}{16}$$

5) Evaluate $\int_C \frac{z\cos z}{\left(z-\frac{\pi}{2}\right)^3} dz$ where c is $|z-1| = 1$

Solution:

Let $f(z) = \frac{z\cos z}{\left(z-\frac{\pi}{2}\right)^3}$

$z = \frac{\pi}{2}$ is a pole of order 3 of the function $f(z)$ and it lies within the given circle

$$\therefore [Res\, f(z)]_{z=\frac{\pi}{2}} = \frac{1}{(3-1)!} \lim_{z\to\frac{\pi}{2}} \left[\frac{d^{3-1}}{dz^{3-1}} \left\{\left(z-\frac{\pi}{2}\right)^3 f(z)\right\}\right]$$

$$= \frac{1}{2}\lim_{z\to\frac{\pi}{2}} \left[\frac{d^2}{dz^2}\left\{\left(z-\frac{\pi}{2}\right)^3 \frac{z\cos z}{\left(z-\frac{\pi}{2}\right)^3}\right\}\right]$$

$$= \frac{1}{2} \lim_{z \to \frac{\pi}{2}} \left[\frac{d^2}{dz^2} (z \cos z) \right] = \frac{1}{2} \lim_{z \to \frac{\pi}{2}} \left[\frac{d}{dz} (\cos z - z \sin z) \right]$$

$$= \frac{1}{2} \lim_{z \to \frac{\pi}{2}} [- \sin z - (\sin z + z \cos z)] = \frac{1}{2} \lim_{z \to \frac{\pi}{2}} [-2 \sin z - z \cos z]$$

$$= \frac{-1}{2} \left[2 \sin\frac{\pi}{2} + \frac{\pi}{2} \cos\frac{\pi}{2}\right] = -\frac{1}{2} \left[2(1) + \frac{\pi}{2} (0)\right] = -\frac{1}{2} (2) = -1$$

∴ By residue theorem

$$\int_C \frac{z \cos z}{\left(z - \frac{\pi}{2}\right)^3} dz =$$

$2\pi i$ ×Sum of the residues of $f(z)$ at the interior poles

$$= 2\pi i\ (-1) = -2\pi i$$

7. Evaluate $\int_C \frac{\cot hz}{z-i} dz$ where C is $|z| = 2$

Solution: Let $f(z)$

$$= \frac{\coth z}{z - i} = \frac{\cos h\, z}{(z - i) \sin h\, z}$$

The poles of $f(z)$are given by $(z - i) \sin h\, z = 0$

$i.e. z = i \pm n \pi i, n$ being zero or an integer. Thus out of the many poles z=i and z=0 are the only poles lying inside the given circle $|z| = 2$. Hence it is enough if we calculate the corresponding residues

$$[Res\ f(z)]_{z=i} = \lim_{z \to i} [(z - i) f(z)]$$

$$\lim_{z \to i} \left[(z - i) \frac{\cos h\, z}{(z - i) \sin h\, z}\right] = \lim_{z \to i} \cot hz = \coth i$$

$$[Res\ f(z)]_{z=0} = \frac{\emptyset (0)}{\Psi (0)}$$

Where $f(z) = \frac{\cos h\, z}{(z-i) \sin h\, z} = \frac{\emptyset (z)}{\Psi (z)}$

$$\left[\frac{\cos hz}{(z - i) \cosh z + \sin h\, z}\right]_{z=0} = -\frac{1}{i}$$

∴ By residue's theorem

$$\int_C \frac{\coth z}{z-i} dz = 2\pi i \times \text{ sum of the residues at z=i and z=0} = 2\pi i \left(\coth i - \frac{1}{i}\right)$$

UNIT III : BILINEAR TRANSFORMATIONS

Evaluation of Real Definite Integrals by Contour Integration

To evaluate certain types of real definite integrals which often arise in solving physical problems Cauchy's Residue theorem is applied as it is simpler than the usual methods of integration. Contour integration is also another method of evaluating a definite integral by making the path of integration about a suitable contour (curve) in the complex plane.

Contour integration is used to evaluate the following types of integrals

1) TYPE-1

To evaluate integral of the type $\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$

2) TYPE-2

Integrals of the type $\int_{-\infty}^{\infty} f(x)dx$

Type-1: Integral of the type $\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$

To evaluate Consider the evaluation of the integrals of the type

$\int_0^{2\pi} F(Cos\,\theta, \sin\theta)d\theta$, where F is a real rational function of $\sin\theta$ and $\cos\theta$

Let $z = e^{i\theta}$ so that $\theta = \frac{\log z}{i}$ and $d\theta = \frac{dz}{iz}$.

But we know that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2} \text{ and } |z| = |e^{i\theta}| = 1.$$

$$\therefore \int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta = \int_c F\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right]\frac{dz}{iz} = \int_c f(z)\, dz\, (say)$$

Where C is the unit circle $|z| = 1$.

Since is a rational function therefore by residue theorem

$\int_c f(z)dz = 2\pi i\ \times$ (Sum of residues of at its poles inside 'C')

Solved Problems

1. Show by the method of residues

$$\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\pi}{\sqrt{a^2 - b^2}} \quad (a > b > 0)$$

Solution: Consider

$$\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} \qquad \rightarrow (1)$$

Let C be the circle given as $|z| = 1$

Put $z = e^{i\theta}$. So that $d\theta = \frac{dz}{iz}$ and

$$\cos\theta = \frac{z + \frac{1}{z}}{2} = \frac{z^2 + 1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \int_c \frac{1}{a + b\left[\frac{z^2+1}{2z}\right]} \cdot \frac{dz}{iz}$$

$$= \frac{1}{i}\int_c \frac{2}{bz^2 + 2az + b}dz = \frac{2}{i}\int_c \frac{dz}{bz^2 + 2az + b} = \frac{2}{i}\int_c f(z)dz$$

Now the poles of $f(z)$ are the roots of $bz^2 + 2az + b = 0$, So

$z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$ are the poles

Let $\alpha = \frac{-a + \sqrt{a^2 - b^2}}{b}$, $\beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$

Since $a > b > 0$,

We have $|\beta| > 1$.

But the product of the roots is 1

$i.e.,$ $|\alpha\beta| > 1$ So that $|\alpha| < 1$

Thus $z = \alpha$ is the only simple pole lies inside C and so

$$f(z) = \frac{1}{b(z-\alpha)(z-\beta)}$$

$$\therefore Res\,(f : z = \alpha) = [Res\, f(z)]_{z=\alpha}\ \underset{z\to\alpha}{\mathrm{Lt}}\,(z-\alpha) f(z)$$

$$\underset{z\to\alpha}{\mathrm{Lt}}\,(z-\alpha)\frac{2}{i}\frac{1}{b(z-\alpha)(z-\beta)} = \frac{2}{i}\ \underset{z\to\alpha}{\mathrm{Lt}}\,\frac{1}{b(z-\beta)}$$

$$= \frac{2}{i}\frac{1}{b(\alpha-\beta)} = \frac{2}{i}\frac{1}{b.\left(\frac{2\sqrt{a^2-b^2}}{b}\right)} = \frac{1}{i\sqrt{a^2-b^2}}$$

$\int_c f(z)dz = 2\pi i \times$ (Sum of residues of at its poles inside 'C')

therefore $\int_c f(z)dz = 2\pi i \times \frac{1}{2}\left(\frac{1}{i\sqrt{a^2-b^2}}\right) = \frac{\pi}{\sqrt{a^2-b^2}}$

2. Show that $\int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{\pi a}{(a^2-b^2)^{\frac{3}{2}}}$. $a > b > 0$

Solution:

Consider $\int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$

Put $z = e^{i\theta}$ So that $d\theta = \frac{dz}{iz}$

where C is $|z| = 1$

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \int_c \frac{1}{\left[a+b\left(\frac{z+\frac{1}{z}}{2}\right)\right]^2}\cdot\frac{dz}{iz}$$

where C is $|z| = 1$

$$= \frac{1}{i}\int_c \frac{4z^2}{(bz^2+2az+b)^2}\cdot\frac{1}{z}\,dz = \frac{4}{i}\int_c \frac{z}{b^2\left(z^2+\frac{2a}{b}z+1\right)^2}\,.dz = \frac{4}{ib^2}\int_c f(z)dz$$

Where

$$f(z) = \frac{z}{\left(z^2+\frac{2a}{b}z+1\right)^2}$$

Here the poles of $f(z)$ are of order 2)lies within the circle C

which are given by

$$z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

But

$$\frac{-a+\sqrt{a^2-b^2}}{b}$$

is the only pole (of order 2) Let this pole represented by α and another by β we get

$$\propto = \frac{-a+\sqrt{a^2-b^2}}{b}, \qquad \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$$

$$[Res\, f(z)]_{z=\alpha} = \frac{1}{(2-1)!}\ \underset{z\to\alpha}{\mathrm{Lt}}\frac{d^{2-1}}{dz^{2-1}}\{(z-\propto)^2\ f(z)\}$$

$$= \underset{z\to\alpha}{\mathrm{Lt}}\frac{d}{dz}\left\{\frac{z}{(z-\beta)^2}\right\} = \underset{z\to\alpha}{\mathrm{Lt}}\frac{(z-\beta)^2.1 - z.2(z-\beta)}{(z-\beta)^4}$$

$$= \underset{z\to\alpha}{\mathrm{Lt}}\frac{-(z+\beta)}{(z-\beta)^3} = \frac{-(\alpha+\beta)}{(\alpha-\beta)^3} = \frac{-\left(\frac{2a}{b}\right)}{\left(\frac{2\sqrt{a^2-b^2}}{b}\right)^3} = \frac{ab^2}{4(a^2-b^2)^{\frac{3}{2}}}$$

∴By Residue theorem we have

$$\int_0^{\pi}\frac{d\theta}{(a+b\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi}\frac{d\theta}{(a+b\cos\theta)^2} = \frac{1}{2}\cdot\frac{4}{ib^2}\int_c f(z)dz$$

$$= \frac{2}{ib^2}\,2\pi i\,\frac{ab^2}{4\,(a^2-b^2)^{\frac{3}{2}}} = \frac{\pi a}{(a^2-b^2)^{\frac{3}{2}}}$$

3) Use method of contour integration to prove that

$$\int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta} = \frac{2a\pi}{1-a^2}, 0 < a < 1$$

Solution:

Let

$z = e^{i\theta}$

and C be the unit circle $|z| = 1$

Then

$dz = e^{i\theta}$ or $d\theta = \frac{dz}{iz}$

and $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}$

The given integral I is $= \int_C \frac{1}{1+a^2-2a\frac{z^2+1}{2z}} \cdot \frac{dz}{iz} = -\frac{1}{i}\int_C \frac{dz}{az^2-(1+a^2)z+a}$

The poles of the integrand are given by

$$\frac{(1+a^2) \pm \sqrt{(1-a^2)^2}}{2a} \quad i.e. \frac{1}{a}$$

and a

Of these poles only $z_0 = a$ lies inside C ($\because a < 1$), Residues at $z = a$ is

$$= \underset{z\to a}{\text{Lt}}(z-a) = \frac{1}{\left(z-\frac{1}{a}\right)(z-a)} = \underset{z\to a}{\text{Lt}} \frac{1}{z-\frac{1}{a}} = \frac{1}{a-\frac{1}{a}} = \frac{a}{a^2-1}$$

By residue theorem

$$I = \frac{-1}{i}.2\pi i.\frac{a}{a^2-1} = \frac{2a\pi}{1-a^2}$$

UNIT III : EVALUATION OF INTEGRALS & BILINEAR TRANSFORMATIONS

Integrals of the type $\int_{-\infty}^{\infty} f(x)dx$

Integration around semi – circle: To solve these types of integrals we consider

$\int_C f(z)\,dz$: where C is the closed contour consisting of the semi Circle C_R : $|z| = R$, together with the real axis from $-R$ to R. If there are no poles of f(z) on the real axis, the circle $|z| = R$, which is arbitrary can be taken such that there is singularity on its circumference C_R in the upper half of the plane, but some poles may lie inside the contour C as specified above.

If $f(z)$ has no singular point on the real axis, by Residue theorem, we have.

$\int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i$ × Sum of residue at interior poles

So we find the value of

$\int_{-\infty}^{\infty} f(x)dx$, Provided $\int_{C_R} |f(z)|dz \to 0$

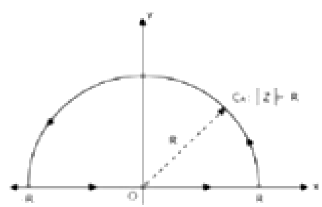
making $R \to \infty$

Solved Problems

1) Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$

Solution: Here Since

$\frac{1}{(x^2+a^2)^2}$ is an even function of x we have



$$\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$$

Here we consider

$$\int_C \frac{dz}{(z^2+a^2)^2} = \int_C f(z)dz$$

Where C is the contour consisting of the semi – circle C_R of radius R together with the part of the real axis from – R to R. The integrand has two poles of order 2 at

$z = ia$ and $z = -ia$.

But $z = ai$ only lies inside the semi-circle of the contour C

∴By residues theorem

$\int_C f(z)dz = 2\pi i$ ×Sum of residues$= 2\pi i \times \{Res\, f(z)\}_{z=ai}$

$$= 2\pi i \left[\underset{z\to ai}{\text{Lt}} \frac{d}{dz} (z-ai)^2 \frac{1}{(z-ai)^2(z+ai)^2} \right]$$

$$= 2\pi i \times \underset{z\to ai}{\text{Lt}} \frac{d}{dz}\left[\frac{1}{(z+ai)^2}\right] = 2\pi i \times \underset{z\to ai}{\text{Lt}} \frac{-2}{(z+ai)^3}$$

$$= 2\pi i \times \underset{z\to ai}{\text{Lt}} \frac{-2}{(2ai)^3} = \frac{\pi}{2a^3}$$

$$i.e., \quad \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \frac{\pi}{2a^3}$$

$$i.e., \quad \int_{-R}^{R} \frac{dx}{(x^2+a^2)^2} + \int_{C_R} \frac{dz}{(z^2+a^2)^2} = \frac{\pi}{2a^3} \qquad \rightarrow (2)$$

$$Now \left| \int_{C_R} \frac{dz}{(z^2+a^2)^2} \right| \le \int_{C_R} \frac{|dz|}{|(z^2+a^2)^2|}$$

$$\le \frac{1}{(R^2-a^2)^2} + \int^{\pi} Rd\theta \quad [\because |z^2+a^2| > |z|^2 - |-a|^2 \text{ and } z = Re^{i\theta}, |dz| = Rd\theta]$$

$$= \frac{R\pi}{(R^2 - a^2)^2}$$

and this $\to 0 \; as \; R \to \infty$

$$\therefore \int_{C_R} \frac{dz}{(z^2 + a^2)} \to \frac{\pi}{2a^3} \; as \; R \to \infty$$

Hence by making $R \to \infty$, equation (2) becomes

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{2a^3}$$

$$\int_{0}^{\infty} \frac{dx}{(x^2 + a^2)^2} = \frac{1}{2}\left(\frac{\pi}{2a^3}\right), by \; (1) = \frac{\pi}{4a^3}$$

2) Using the method of contour integration prove that

$$\int_{0}^{\infty} \frac{dx}{x^6 + 1} = \frac{\pi}{3}$$

Solution: Here Since the integrand is an even function, we have

$$\int_{-\infty}^{\infty} \frac{dx}{x^6 + 1} = \frac{1}{2}\int_{0}^{\infty} \frac{dx}{x^6 + 1} \quad consider \quad \int_{C} \frac{dz}{z^6 + 1} = \int_{C} f(z)dz$$

Where C is the contour consisting of the semi-circle C_R

of radius R together with real axis from $-R \; to + R$

The poles of $f(z) = \frac{1}{z^6+1}$ are the roots of the equation $z^6 + 1 = 0$

i.e. $z^6+1=0 \implies z=(-1)^{\frac{1}{6}}$

$\therefore z=(\cos\pi+i\sin\pi)^{\frac{1}{6}}$

$$=[\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)]^{\frac{1}{6}}=\cos\frac{(2n+1)\pi}{6}+i\sin\frac{(2n+1)\pi}{6}$$

Demoiver's theorem

Where $n=0,1,2,3,4,5 \quad or \quad z=e^{\frac{(2n+1)\pi i}{6}}$

Where $n=0,1,2,3,4\,5$ Or $z=e^{i\pi/6},\ e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}$

Of these poles only $z=e^{\frac{\pi i}{6}}, z=e^{\frac{3\pi i}{6}}, z=e^{\frac{5\pi i}{6}}$, lies inside semi-circle

$$\therefore Res\,[f(z)]_{z=e^{\frac{\pi i}{6}}}=\underset{z\to e^{\frac{\pi i}{6}}}{\mathrm{Lt}}\left[\left(z-e^{\frac{\pi i}{6}}\right).\frac{1}{z^6+1}\right]\left(=\frac{0}{0}\right)$$

$\underset{z\to e^{\frac{\pi i}{6}}}{\mathrm{Lt}}\left[\frac{1}{6z^5}\right]$ By L Hospital's Rule $=\frac{1}{6}\,e^{-\frac{5\pi i}{6}}$

Similarly $Res\,[f(z)]_{z=e^{\frac{3\pi i}{6}}}=\frac{1}{6}\,e^{-\frac{5\pi i}{2}}$ and $Res\,[f(z)]_{z=e^{\frac{5\pi i}{6}}}=\frac{1}{6}\,e^{-\frac{25\pi i}{6}}$

Hence by residue theorem

$\int_C f(z)dz=2\pi i\ \times$ Sum of the residue at the poles with in 'C'

$$=\frac{2\pi i}{6}\left[e^{-\frac{5\pi i}{6}}+e^{-\frac{5\pi i}{2}}+e^{-\frac{25\pi i}{6}}\right]$$

$$=\frac{\pi i}{3}\left[\left(\cos\frac{5\pi}{6}-i\sin\frac{5\pi}{6}\right)+\left(\cos\frac{5\pi}{2}-i\sin\frac{5\pi}{2}\right)+\left(\cos\frac{25\pi}{6}-i\sin\frac{25\pi}{6}\right)\right]=\frac{2\pi}{3}$$

$$i.e.,\ \int_{-R}^{R}f(x)dx+\int_{C_R}f(z)dz=\frac{2\pi}{3}$$

But $\int_{C_R}\frac{dz}{z^6+1}\to 0\ as\ z=Re^{i\theta}$ and $R\to\infty$

Hence $\int_{-\infty}^{\infty}f(x)dx=\frac{2\pi}{3}$, $i.e.\ \int_{-\infty}^{\infty}\frac{dx}{x^6+1}=\frac{2\pi}{3}\ or\ \int_0^{\infty}\frac{dx}{x^6+1}=\frac{\pi}{3}$

3) Evaluate by contour integration $\int_0^\infty \frac{dx}{1+x^2}$

Solution:

Let

$$\int_C \frac{dx}{z^2+1} = \int_C f(z)dz$$

where C is the contour consisting of semi-circle

C_R of radius R together with the part of the real axis form – R to R. The integrand has simple poles at $z = \pm i$.

The pole $z = i$ is inside C and $z = -i$ is outside C

By Cauchy Residue theorem

$$\int_C f(z)dz = 2\pi i\left(\frac{1}{2i}\right) = \pi \ i.e. \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \pi \qquad \rightarrow (1)$$

Hence by making $R \rightarrow \infty$,

Relation (1) becomes

$$\int_{-\infty}^{\infty} f(x)dx + \underset{z\rightarrow\infty}{\mathrm{Lt}} \int_{C_R} f(z)dz = \pi$$

As $R \rightarrow \infty$ for any point on the semi-circle C_R $|z| \rightarrow \infty \ i.e. f(z) \rightarrow 0$

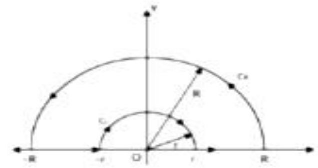
$Thus \int_{-\infty}^{\infty} f(x)dx = \pi \ or \ \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \pi \ or\ 2\int_0^\infty \frac{dx}{1+x^2} = \pi$ Since the integrand is an even function. Therefore

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

4. Evaluate $\int_0^{\infty} \frac{\sin mx}{x} dx$ When m>0using residue theorem

Solution: We know that sin mx is the imaginary part of e^{imx}.

Therefore, consider the function $f(z) = \frac{e^{imz}}{z}$.



Now $f(z)$ has a simple pole at $z = 0$ which lies on the real axis. This pole will be avoided by indentation. For this draw a small semi – circle $C_r : |z| = r$ which contours the singular point z=0 inside a semi – circle C_R: $|z| = R$

Now evaluate the integral $\int_C f(z)dz$, where C consists of parts of the real axis r to R, the semi – circle real axis '–R 'to '–r' and a small semi-circle C_r Since C is the closed contour has no singularity, we have by Cauchy's theorem.

$$\int_C f(z)dz = 0$$

$$i.e. \int_r^R f(x)dx + \int_{C_R} f(z)dz + \int_{-R}^{-r} f(x)dx + \int_{C_r} f(z)dz = 0 \qquad \rightarrow (1)$$

Consider

$$\int_{C_R} f(z)dz = \int_0^{\pi} \frac{e^{im(Re)}}{Re^{i\theta}} . Rie^{i\theta}\, d\theta \quad \begin{bmatrix} z = Re^{i\theta} \\ dz = Rie^{i\theta}\, d\theta \end{bmatrix}$$ Put $z = e^{i\theta}$ then $d\theta = \frac{dz}{iz}$

$$= i \int_0^{\pi} e^{imR\cos\theta(\cos\theta - mR\sin\theta)}\, d\theta$$

But $\left|e^{imR\cos\theta(\cos\theta - mR\sin\theta)}\right| = \left|e^{imR\cos\theta(\cos\theta - mR\sin\theta)}\right| = e^{-mR\,\mathrm{Sin}\,\theta}$

$$\therefore \left|\int_{C_R} f(z)dz\right| \leq \int_0^{\pi} e^{-mR\sin\theta}\, d\theta$$

Note that when θ

increase from $0\ to\ \frac{\pi}{2}$, $\frac{\sin\theta}{\theta}$ decreases from $1\ to\ \frac{2}{\pi}$,

$i.e. for\ 0 \leq \theta \leq \frac{\pi}{2}, \frac{\sin\theta}{\theta} \geq \frac{2}{\pi} \sin\theta \geq \frac{2\theta}{\pi}$

$$\therefore \left|\int_C f(z)dz\right| \leq 2\int_0^{\pi/2} e^{\frac{-2mR\theta}{\pi}} d\theta = \left[\frac{-\pi}{mR} e^{\frac{-2mR\theta}{\pi}}\right]_0^{\pi/2} = \frac{\pi}{mR}(1 - e^{-mR})$$

Which tends to $0\ as\ R \to \infty$

Also $\int_C f(z)dz = i \int_{\pi}^{0} e^{imR(cos\theta - mR\ sin\theta)} d\ \theta$

Which tends to $i \int_{\pi}^{0} d\theta = -i\pi\ as\ r \to 0$

Thus , when $R \to \infty$ and when $r \to 0$ from (1) we have

$$\int_0^{\infty} f(x)dx + 0 + \int_{-\infty}^{0} f(x)dx + (-i\pi) = 0$$

Which gives $\int_{-\infty}^{\infty} f(x)dx = i\pi\ \ or\ \int_{-\infty}^{\infty} \frac{e^{imx}}{x} dx = i\pi \qquad \to (2)$

Equating the imaginary parts of both sides, we get

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x} dx = \pi \ , \qquad Hence \int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}$$

Note: Equating the real parts on both side in (2) we get

$$\int_{-\infty}^{\infty} \frac{cosmx}{x} dx = 0$$

Mapping or Transformation

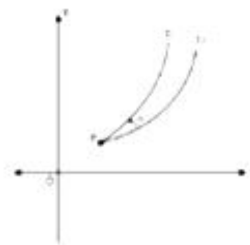
Transformation of z-plane to w-plane by a function

- If $y = f(x)$is a real valued function then the equation
- $y = f(x)$ gives a relationship between the points on x –axis and points on y –axis.

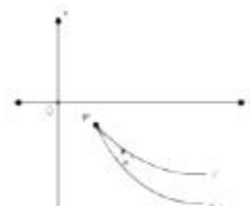
This relationship can be represented by a curve drawn in xy - plane. But when $f(z)$ is complex valued function of a complex variable z, no such convenient graphical representation is possible, because z and w require two different planes to represent them. However by representing $z = x + iy$ in z – plane and the corresponding $w = u + iv$ in w-plane we can establish a graphical corresponding between two planes

Conformal Transformation

Definition:



- Suppose under the transformation $w = f(z)$ i.e $u = u(x,y)$ and $v = v(x,y)$ the point $P(x_0, y_0)$ of the Z-Plane is mapped into the point $P(x_0, y_0)$.
- Suppose the mapping takes C_1 and C_2 in to the curves C_1 and C_2 which are intersected at $P(u_0, v_0)$.
- If the transformation is such that the angles between C_1 and C_2at (x_0, y_0) is equal both in magnitude and sense to the angle between C_1 and C_2at (x_0, y_0) then it is said to be Conformal at (x_0, y_0).



- If the transformation preserves the magnitudes but not necessarily sense, then it is

called **isogonal.**

Bilinear Transformation

Definition:

The transformation $w = \frac{az+b}{cz+d}$ where a,b,c,d are complex constants and $ad - bc \neq 0$ is known as" bilinear transformation

Properties of Bilinear Transformations

1. A bilinear transformation is conformal:

We have $w = \frac{az+b}{cz+d}$

Differentiating $w = \frac{(cz+d)(z)-(az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$

Since $ad - bc \neq 0$

We have $w' \neq 0$ is nowhere zero

Hence the mapping defined above is conformal.

2) Every bilinear transformation maps the totality of circles and straight lines in the z – plane onto the totality of circles and straight lines in the w – plane.

Proof: Let $w = \frac{az+b}{cz+d}$ *is* the conformal mapping, which can be written as

$$w = \frac{az+b}{cz+d} = \frac{c(az+b)}{c(cz+d)} = \frac{acz+bc)}{c(cz+d)} = \frac{acz+ad-ad+bc)}{c(cz+d)} = \frac{a(cz+d)-ad+bc)}{c(cz+d)}$$

$$w = \frac{a}{c} + \frac{bc-ad}{a^2}\frac{1}{\left(z+\frac{d}{c}\right)}$$

This is a combination of the transformation $w_1 = z + \frac{d}{c}, w_2 = \frac{1}{w_1}, w_3 = \frac{bc-ad}{c^2}w_2$ and $w = \frac{a}{c} + w_3$ by these transformations we pass from Z plane to w_1 plane then to w_3 plane. These transformations are the standard transformation $w = z + c, w = cz$ and $w = \frac{1}{z}$.

We know that these transformations map the totality of circle and straight lines in the z – plane onto the totality of circles and straight lines in the w-plane.

This is a combination of the transformation $w_1 = z + \frac{d}{c}, w_2 = \frac{1}{w_1}, w_3 = \frac{bc-ad}{c^2} w_2$ and

$w = \frac{a}{c} + w_3$ by these transformations we pass from Z plane to w_1 plane then to w_3 plane. These transformations are the standard transformation $w = z + c, w = cz$ and $w = \frac{1}{z}$.

We know that these transformations map the totality of circle and straight lines in the z – plane onto the totality of circles and straight lines in the w-plane.

Similarly we can write $w_3 - w_4,\ w_1 - w_4, w_3 - w_2$

similarly defining relations for other points we get

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

This proves the result.

7) Find the bilinear transformation that maps the points $(\infty, i, 0)$ into the points $(0, i, \infty)$

Solution:

Let $z_1 = \infty, z_2 = i, z_3 = 0$ and $w_1 = 0, w_2 = i, w_3 = \infty$

Let $z_4 = z$ and $w_4 = w$ so that z and w are a pair of general points.

Substituting these, the required bilinear transformation is

$$\frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)} = \frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)}$$

$$= \frac{(\infty - i)(0 - z)}{(\infty - z)(0 - i)} = \frac{(0 - i)(\infty - w)}{(0 - w)(\infty - i)}$$

Here $\frac{\infty - w}{\infty - i}$ is taken as 1 and $\frac{\infty - i}{\infty - z}$ is taken as 1

$$\therefore \frac{z}{i} = \frac{i}{w} \Rightarrow w = \frac{i^2}{z} = \frac{-1}{z}$$

8) Find the bilinear transformation which transform the points $\infty, i, 0$ in the z-plane into $0, i, \infty$. in the w-plane

Solution: Let $z_1 = \infty, z_2 = i, z_3 = 0 \text{ and } w_1 = 0, w_2 = i, w_3 = \infty$

The required transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$= \frac{(w-0)(i-\infty)}{(0-i)(\infty-w)} = \frac{(z-\infty)(i-0)}{(\infty-i)(0-z)} = \left(\frac{-w}{i}\right)\left(\frac{i-\infty}{\infty-w}\right) = \left(\frac{-i}{z}\right)\left(\frac{z-\infty}{\infty-i}\right)$$

$$or \left(\frac{-w}{i}\right)\lim_{n\to\infty}\left(\frac{i-n}{n-w}\right) = \left(\frac{-i}{z}\right)\lim_{n\to\infty}\left(\frac{z-n}{n-i}\right) \text{ form} \frac{\infty}{\infty}$$

$$or \left(\frac{-w}{i}\right)\lim_{n\to\infty}\left(\frac{0-1}{1-0}\right) = \left(\frac{-i}{z}\right)\lim_{n\to\infty}\left(\frac{0-1}{1-0}\right) \text{ L'Hospital's rule}$$

$$or \left(\frac{-w}{i}\right)(-1) = \left(\frac{-i}{z}\right)(-1) \, or \, \frac{w}{i} = \frac{i}{z} \, or \, w = \frac{-1}{z}$$

9) Find the bilinear transformation which maps the points $\infty, i, 0$ in the z-plane into $-1, -i, 1$ in the w-plane

Solution: Let $z_1 = \infty, z_2 = i, z_3 = 0 \text{ and } w_1 = -1, w_2 = -i, w_3 = 1$

The required transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} or \frac{(w+1)(-i-1)}{(-1+i)(1-w)} = \frac{(z-\infty)(i-0)}{(\infty-i)(0-z)} = \left(\frac{-i}{z}\right)\lim_{n\to\infty}\left(\frac{z-n}{n-i}\right)$$

$$or \frac{(-1)(w+1)(1+i)}{(-1+i)(1-w)} = \left(\frac{-i}{z}\right)\left(\frac{0-1}{1-0}\right) = \left(\frac{-i}{z}\right)(-1) = \frac{i}{z}$$ (applying L' Hospital's rule)

$$or \frac{1+w}{1-w} = \frac{-i(-1+i)}{(1+i)z} = \frac{1}{z}$$

Applying componendo and dividendo

$$\frac{(1+w)+(1-w)}{(1+w)-(1-w)} = \frac{1+z}{1-z} or \frac{2}{2w} = \frac{1+z}{1-z} or \frac{1}{w} = \frac{1+z}{1-z}$$

Therefore $w = \frac{1-z}{1+z}$

10) Find the bilinear transformation which maps the points $(-1, 0, 1)$ into the points $(0, i, 3i)$

Solution: Given $z_1 = -1, z_2 = 0, z_3 = 1$ *and* $w_1 = 0, w_2 = i, w_3 = 3i$

We know that

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

$$\Rightarrow \frac{(w - 0)(i - 3i)}{(0 - i)(3i - w)} = \frac{(z + 1)(0 - 1)}{(-1 - 0)(1 - z)} \Rightarrow \frac{2w}{3i - w} = \frac{z + 1}{1 - z}$$

$$\Rightarrow (2w)(1 - z) = (z + 1)(3i - w)$$

$$\Rightarrow (2w)(1 - z) = 3iz + 3i - wz - w$$

$$\Rightarrow w(2 - 2z + z + 1) = 3i\,(z + 1)$$

$$\Rightarrow w(-z + 3) = 3i(z + 1) \Rightarrow w = \frac{3i(z + 1)}{(3 - z)}$$

11) Determine the bilinear transformation that maps the points $(1 - 2i,\ 2 + i, 2 + 3i)$ into the points $(2 + i, 1 + 3i, 4)$

Solution: Given $z_1 = 1 - 2i, z_2 = 2 + i, z_3 = 2 + 3i$ and

$w_1 = 2 + i, w_2 = 1 + 3i, w_3 = 4$

Let $z_4 = z$ and $w_4 = w$ so that z and w are a pair of general points substituting these in

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)}$$

We get

$$\frac{(1 - 2i - 2 - i)(2 + 3i - z)}{(1 - 2i - z)(2 + 3i - 2 - i)} = \frac{(2 + i - 1 - 3i)(4 - w)}{(2 + i - w)(4 - 1 - 3i)}$$

$$i.e. \frac{2 + 3i - z}{1 - 2i - z} = -\frac{1}{3}\left(\frac{4 - w}{2 + i - w}\right) = \frac{w - 4}{3(2 + i - w)}$$

$$w = \frac{z(10 + 3i) - 7 - 6i}{4z - 7 - 7i}$$

12) Find the bilinear transformation which maps the points $(-i, 0, i)$ into the points $(-1, i, 1)$ respectively.

Solution: $z_1 = -i, z_2 = 0, z_3 = i$ and $w_1 = -1, w_2 = i, w_3 = 1$

The required transformation is given by

$$\frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)}$$

Substituting the values we get

$$\frac{(-1 - i)(1 - w)}{(-1 - w)(1 - i)} = \frac{(-i - 0)(i - z)}{(-i - z)(i - 0)}$$

$$i.e \frac{(1 + i)(1 - w)}{(1 - i)(i + w)} = \frac{i(i - z)}{i(i + z)} = \frac{i - z}{i + z}$$

$$i.e. \frac{1 - w}{1 + w} = \frac{i - z}{i + z} \cdot \frac{1 - i}{1 + i} = \frac{i - z}{i + z}(-i) = \frac{zi + 1}{z + i}$$

Applying componendo and dividendo, we get

$$\frac{(1 - w) + (1 + w)}{(1 - w) - (1 + w)} = \frac{(zi + 1) + (z + i)}{(zi + 1) - (z + i)} = \frac{(1 + i)(z + 1)}{(1 - i)(-z + 1)}$$

$$i.e. \frac{2}{-2w} = \frac{i(z + 1)}{(-z + 1)} \quad or \; -w = \frac{-z + 1}{i(z + 1)} = \frac{i(-z + 1)}{i^2(z + 1)}$$

$w = \frac{-iz + i}{z + 1}$ .is the required transformation

UNIT IV : Fourier Series and Fourier Transforms

Suppose that a given function $f(x)$ defined in $[-\pi, \pi]$ (or) $[0, 2\pi]$ (or) in any other interval can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

The above series is known as the Fourier series for $f(x)$ and the constants $a_0, a_n, b_n\,(n = 1, 2, 3 - - - - -)$ are called Fourier coefficients of $f(x)$

Periodic Functions:-

A function $f(x)$ is said to be periodic with period $T > 0$ if for all x $f(x+T) = f(x)$ and T is the least of such values

Example:- $\sin x = \sin(x + 2\pi) = \sin(x + 4\pi) = - - - - -$ the function $\sin x$ is periodic with period 2π there is no positive value T, $0 < T < 2\pi$ such that $\sin(x+T) = \sin x \forall\ x$

Euler's Formula:-

The Fourier series for the function $f(x)$ in the interval $c \le x \le c + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx.dx \text{ and}$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx.dx$$

These values of a_0, a_n, b_n are known as Euler's formula

Corollary:- if $f(x)$ is to be expanded as a fourier series in the interval $0 \le x \le 2\pi$, put $c = 0$ then the formulae (1) reduces to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx.dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx.dx$$

Corollay 2:- if $f(x)$ is to expanded as a fourier series in $[-\pi, \pi]$ put $c = -\pi$, the interval becomes $-\pi \le x \le \pi$ and the formula (1) reduces to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$$

Conditions For Fourier Expansion:-

Dirichlet has formulated certain conditions known as Dirichlet conditions under which certain functions posses valid Fourier Expansions.

A given function $f(x)$ has a valid Fourier series expansion of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

Where a_0, a_n, b_n are constants, provided

(i) $f(x)$ is well defined and single – valued except possibly at a finite number of points in the interval of definition

(ii) $f(x)$ has a finite number of discontinuities in the interval of definition

(iii) $f(x)$ has al most a finite number of maxima and minima in the interval of definition

Note:- The above conditions are sufficient but not necessary

Functions Having Points of Discontinuity :-

In Euler's formulae for a_0, a_n, b_n it was assumed that $f(x)$ is continuous. Instead a function may have a finite number of discontinuities. Even then such a function is expressible as a Fouries series

Let $f(x)$ be defined by

$$f(x)=\phi(x) \quad c<x<x_0$$
$$=\phi(x) \quad x_0<x<c+2\pi$$

Where x_0 is the point of discontinuity in $(c,c+2\pi)$ in such cases also we obtain the Fourier series for $f(x)$ in the usual way. The values of a_0, a_n, b_n are given by

$$a_0=\frac{1}{\pi}\left[\int_c^{x_0}\phi(x)dx+\int_{x_0}^{c+2\pi}\phi(x)dx\right]$$

$$a_n=\frac{1}{\pi}\left[\int_c^{x_0}\phi(x)\cos nx.dx+\int_{x_0}^{c+2\pi}\phi(x)\cos nx.dx\right]$$

$$b_n=\frac{1}{\pi}\left[\int_c^{x_0}\phi(x)\sin nx.dx+\int_{x_0}^{c+2\pi}\phi(x)\sin nx.dx\right]$$

Note :-

$$(i)\int_{-\pi}^{\pi}\cos mx\ \cos nx\,dx=\begin{cases}0 \ \ for\ \ m\neq n\\ \pi,\ \ for\ \ m=n>0\\ 2\pi,\ \ for\ \ m=n=0\end{cases}$$

$$(ii)(i)\int_{-\pi}^{\pi}\sin mx\ \sin nx\,dx=\begin{cases}0\ \ for\ \ m\neq n\ \ and\ \ m=n=0\\ \pi,\ \ for\ \ m=n>0\end{cases}$$

Examples:-

1. **Express $f(x)=x-\pi$ as Fourier series in the interval $-\pi<x<\pi$**

Sol Let the function $x-\pi$ be represented by the Fourier series

$$x-\pi=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \rightarrow (1)$$

Then

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)dx$$
$$=\frac{1}{\pi}\left[\int_{-\pi}^{\pi}xdx-\pi\int_{-\pi}^{\pi}dx\right]$$
$$=\frac{1}{\pi}\left[0-\pi.2\int_0^{\pi}dx\right] \quad (\because x\ is\ odd\ \ function)$$
$$=\frac{1}{\pi}\left[-2\pi(x)_0^{\pi}\right]$$
$$=-2(\pi-0)=-2\pi\ and$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx.dx$$
$$=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\cos nx.dx$$
$$=\frac{1}{x}\left[\int_{-\pi}^{\pi}x\cos nx.dx-\pi\int_{-\pi}^{\pi}\cos nx.dx\right]$$
$$=\frac{1}{\pi}\left[0-2\pi\int_0^{\pi}\cos nx.dx\right]$$

$$\therefore a_n = -2\int_0^{\pi} \cos nx.dx$$

$$= -2\left(\frac{\sin nx}{n}\right)_0^{\pi}$$

$$= \frac{-2}{n}(\sin n\pi - \sin 0)$$

$$= \frac{-2}{n}(0-0) = 0 \quad for \;\; n = 1, 2, 3.........$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\sin nx.dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx - \pi\int_{-\pi}^{\pi}\sin nx.dx\right]$$

$$= \frac{1}{\pi}\left[2\int_0^{\pi} x\sin nx.dx - \pi(0)\right]$$

($\because x\cos nx$ is odd function and $\cos nx$ is even function)

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n}+0\right) - (0+0)\right]$$

$$= \frac{-2}{\pi}\cos n\pi = \frac{-2}{n}(-1)^n$$

$$= \frac{2}{n}(-1)^{n+1}\ \forall n = 1, 2, 3.....$$

Substituting the values of a_0, a_n, b_n in (1),

We get

$$x - \pi = -\pi + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{\pi}\sin nx$$

$$= -\pi + 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + ...\right]$$

2. Find the Fourier series to represent the function e^{-ax} $from \;\; x = -\pi \;\; to \;\; \pi$. Deduce from this that

$$\frac{\pi}{\sinh\pi} = 2\left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - - - - - -\right]$$

Sol. Let the function e^{-ax} be represented by the Fourier series

$$e^{-ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \to (1)$$

Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx = \frac{1}{\pi}\left(\frac{e^{-ax}}{-a}\right)_{-\pi}^{\pi}$$

$$= \frac{-1}{a\pi}\left(e^{-a\pi} - e^{a\pi}\right) = \frac{e^{a\pi} - e^{-a\pi}}{a\pi}$$

$$\therefore \frac{a_0}{2} = \left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right]\frac{1}{a\pi} = \frac{\sinh a\pi}{a\pi}$$

And

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx.dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left(-a\cos nx + n\sin nx\right)\right]_{-\pi}^{\pi}$$

$$\int e^{ax}\cos bx.dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)$$

$$\therefore a_n = \frac{1}{\pi}\left\{\frac{e^{-ax}}{a^2+n^2}\left(-a\cos n\pi + 0\right) - \frac{e^{a\pi}}{a^2+n^2}\left(-a\cos n\pi + 0\right)\right\}$$

$$= \frac{a}{\pi\left(a^2+n^2\right)}\left(e^{a\pi} - e^{-a\pi}\right)\cos n\pi$$

$$= \frac{2a\cos n\pi \sinh a\pi}{\pi\left(a^2+n^2\right)}$$

$$= \frac{(-1)^n\, 2a\sinh a\pi}{\pi\left(a^2+n^2\right)}\left(\because \cos n\pi = (-1)^n\right)$$

Finally $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx.dx$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left(-a\sin nx - n\cos nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-a\pi}}{a^2+n^2}\left(0 - n\cos n\pi\right) - \frac{e^{a\pi}}{a^2+n^2}\left(0 - n\cos n\pi\right)\right]$$

$$= \frac{n\cos n\pi e^{a\pi} - e^{-a\pi}}{\pi\left(a^2+n^2\right)} = \frac{(-1)^n\, 2n\sinh a\pi}{\pi\left(a^2+n^2\right)}$$

Substituting the values of $\frac{a_0}{2}, a_n$ *and* b_n in (1) we get

$$e^{-ax} = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n\, 2a\sinh a\pi}{\pi\left(a^2+n^2\right)}\cos nx + (-1)^n\, 2n\frac{\sinh a\pi}{\pi\left(a^2+n^2\right)}\sin nx\right]$$

$$= \frac{2\sinh a\pi}{\pi}\left\{\left(\frac{1}{2a} - \frac{a\cos x}{1^2+a^2} + \frac{a\cos 2x}{2^2+a^2} - \frac{a\cos 3x}{3^2+a^2} + - - - - -\right) - \left(\frac{\sin x}{1^2+a^2} - \frac{2\sin 2x}{2^2+a^2} + \frac{3\sin 3x}{3^2+a^2} - - -\right)\right\}$$

Deduction:-

Putting $x = 0$ and $a = 1$ in (2), we get

$$1 = \frac{2\sinh \pi}{\pi}\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - - -\right]$$

$$\frac{\pi}{\sinh \pi} = 2\left(\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - - - -\right)$$

3. Find the Fourier series of the periodic function defined as $f(x) = \begin{bmatrix} -\pi & \pi < x < 0 \\ x & 0 < x < \pi \end{bmatrix}$

Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+------=\frac{\pi^2}{8}$

Sol. Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \to (1)$

Then

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$
$$=\frac{1}{\pi}\left[\int_{-\pi}^{o}(-\pi)dx+\int_0^{\pi}xdx\right]$$
$$=\frac{1}{\pi}\left[-\pi(x)_{-\pi}^{0}+\left(\frac{x^2}{2}\right)_0^{\pi}\right]$$
$$=\frac{1}{\pi}\left[-\pi^2+\frac{\pi^2}{2}\right]=\frac{1}{\pi}\left[\frac{-\pi^2}{2}\right]=\frac{-\pi}{2}$$

$$=\frac{1}{\pi n^2}(\cos n\pi-1)=\frac{1}{\pi n^2}\left[(-1)^{n-1}\right]$$

$$a_1=\frac{-2}{1^2.\pi},a_2=0,a_3=\frac{-2}{3^2.\pi},a_4=0,a_5=\frac{-2}{5^2.\pi},---$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx.dx$$
$$=\frac{1}{\pi}\left[\int_{\pi}^{0}(-\pi)\sin nx.dx+\int_0^{\pi}x\sin nx.dx\right]$$
$$=\frac{1}{\pi}\left[\pi\left(\frac{\cos nx}{n}\right)_{-\pi}^{0}+\left(-x\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right)_0^{\pi}\right]$$
$$=\frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi)-\frac{\pi}{n}\cos n\pi\right]$$
$$=\frac{1}{n}(1-2\cos n\pi)$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx.dx$$
$$=\frac{1}{\pi}\left[\int_{-\pi}^{o}(-\pi)\cos nx.dx+\int_0^{\pi}x\cos nx.dx\right]$$
$$=\frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0}+\left(x\frac{\sin nx}{n}+\frac{\cos nx}{n^2}\right)_0^{\pi}\right]$$
$$=\frac{1}{\pi}\left[0+\frac{1}{n^2}\cos n\pi-\frac{1}{\pi n^2}\right]$$

$b_1=3,b_2=\frac{-1}{2},b_3=1,b_4=\frac{-1}{4}$ and so --- on substituting the values of a_0,a_n *and* b_n in (1), we get

$$f(x)=\frac{-\pi}{4}-\frac{2}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+---\right)+\left(3\sin x-\frac{\sin 2x}{2}+\frac{3\sin 3x}{3}-\frac{\sin 4x}{4}+--\right)$$

Deduction:-

Putting $x=0$ in (2), we obtain

$$f(0)=\frac{-\pi}{4}-\frac{2}{4}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+---\right)$$

Now $f(x)$ is discontinuous at $x=0$

$$f(0-0)=-\pi \text{ and } f(0+0)=0$$

$$f(0)=\frac{1}{2}\left[f(0-0)+f(0+0)\right]=\frac{-\pi}{2}$$

Now (3) becomes

$$\frac{-\pi}{2}=\frac{-\pi}{4}-\frac{2}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+----\right)$$

$$=\frac{\pi^2}{8}$$

Even and Odd Functions:-

A function $f(x)$ is said to be even if $f(-x)=f(x)$ and odd if $f(-x)=-f(x)$

Example:- $x^2, x^4+x^2+1, e^x+e^{-x}$ are even functions

$x^3, x, \sin x, \cos ecx$ are odd functions

Note:-

1. Product of two even (or) two odd functions will be an even function
2. Product of an even function and an odd function will be an odd function

Note 2:- $\int_{-a}^{a} f(x)dx=0$ when $f(x)$ is an odd function

$=2\int_0^a f(x)dx$ when $f(x)$ is even function

Fourier series for even and odd functions

We know that a function $f(x)$ defined in $(-\pi,\pi)$ can be represented by the Fourier series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

Where $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx.dx$$

And $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx.dx$

Case (i):- when $f(x)$ is even function

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}f(x)dx$$

Since $\cos nx$ is an even function, $f(x)\cos nx$ is also an even function

Hence $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$

$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx.dx$

Since $\sin nx$ is an odd function, $f(x)\sin nx$ is an odd function

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx = 0$$

$\therefore$ If a function $f(x)$ is even in $(-\pi,\pi)$, its Fourier series expansion contains only cosine terms

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where $a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx.dx, n = 0,1,2,-----$

Case 2:- when $f(x)$ is an odd function in $(-\pi,\pi)$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0$ since $f(x)$ is odd

Since $\cos nx$ is an even function, $f(x)\cos nx$ is an odd function and hence

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx = 0$$

Since $\sin nx$ is odd function ; $f(x)\sin nx$ is an even function

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx.dx$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx.dx$

Thus, if a function $f(x)$ defined in $(-\pi,\pi)$ is odd, its Fourier expansion contains only sine terms

Examples:-

1. Expand the function $f(x) = x^2$ as a Fourier series in $(-\pi,\pi)$, hence deduce that

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ---- = \frac{\pi^2}{12}$

Sol. Since $f(-x) = (-x)^2 = x^2 = f(x)$

Hence in its Fourier series expansion, the sine terms are absent

$$\therefore x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx$$

$$= \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx.dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx.dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + 2\pi\frac{\cos nx}{n^2} + 2.0\right]$$

$$= \frac{4\cos n\pi}{n^2} = \frac{4}{n^2}(-1)^n$$

Substituting the values of a_0 *and* a_n from (2) and (3) in (1) we get

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \cos nx$$

$$= \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + --- \right) \to (4)$$

<u>Deductions:-</u>

Putting $x = 0$ in (4), we get

$$0 = \frac{\pi^2}{3} - 4\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ---\right)$$

$$\Rightarrow 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ---- = \frac{\pi^2}{12}$$

2. Find the Fourier series to represent the function $f(x) = |\sin x|, -\pi < x < \pi$

Sol Since $|\sin x|$ **is an even function,**

$b_n = 0 \quad for\ all\ n$

Let $f(x) = |\sin x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \to (1)$

Where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| dx$$
$$= \frac{2}{\pi}\int_0^{\pi} \sin x \, dx$$
$$= \frac{2}{\pi}(-\cos x)_0^{\pi}$$
$$= \frac{-2}{\pi}(-1-1) = \frac{4}{\pi} \quad and$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx = \frac{2}{\pi}\int_0^{\pi} \sin x.\cos nx \, dx$$
$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin(1+n)x + \sin(1-n)x\right]dx$$
$$= \frac{1}{\pi}\left[-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi} \, , n \neq 1$$
$$= -\frac{1}{\pi}\left[\frac{\cos(1+n)\pi}{1+n} + \frac{\cos(1-n)\pi}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right]_0^{\pi} \quad n+1$$
$$= \frac{-1}{\pi}\left[\frac{(-1)^{n+1}-1}{1+n} + \frac{(-1)^{n+1}-1}{1-n}\right]$$
$$= \frac{-1}{\pi}\left[(-1)^{n+1}\left\{\frac{1}{1+n}+\frac{1}{1-n}\right\} - \left\{\frac{1}{1+n}+\frac{1}{1-n}\right\}\right]$$
$$= \frac{-1}{\pi}\left[(-1)^{n+1}\frac{2}{1-n^2} - \frac{2}{1-n^2}\right]$$
$$= \frac{2}{\pi(n^2-1)}\left[(-1)^{n+1}-1\right]$$
$$= \frac{-2}{\pi(n^2-1)}\left[1+(-1)^n\right]$$
$$\therefore a_n = \begin{cases} 0 & if \;\; n \; is \; odd \\ \dfrac{-4}{\pi(n^2-1)} & if \;\; n \;\; is \;\; even \end{cases}$$
$$for \;\; n=1, a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x.\cos x \;\; dx$$
$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x \;\; dx$$
$$= \frac{1}{\pi}\left(\frac{-\cos 2x}{2}\right)_0^{\pi}$$
$$= \frac{-1}{\pi}(\cos 2\pi - 1) = 0$$

Substituting the values of $a_0, a_1 \; and \; a_n$ in (1)

We get $|\sin x| = \frac{2}{\pi} + \sum_{n=2,4,--}^{\infty} \frac{-4}{\pi(n^2-1)} \cos nx$

$$= \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2,4---}^{\infty} \frac{\cos nx}{n^2-1}$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1}$$

(replace n by 2n)

Hence $|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + -----\right)$

Half –Range Fourier Series:-

1) The sine series:-

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$where\ b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx.dx$$

2) The cosine series:-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$where\ \ a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx\ and$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx.dx$$

Note:-

1) Suppose $f(x) = x\ in\ [0,\pi]$, it can have Fourier cosine series expansion as well as Fourier sine series expansion in $[0,\pi]$
2) If $f(x) = x^2\ in\ [0,\pi]$, can have Fourier cosine series as well as sine series

Examples:-

1. Find the half range sine series for $f(x) = x(\pi - x)\ in\ 0 < x < \pi$. Deduce that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + ---- = \frac{\pi^3}{32}$

Ans. The Fourier sine series expansion of $f(x)$ in $(0,\pi)$ is

$$f(x) = x(\pi - x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$where\ b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx.dx$$

$$hence\ b_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin nx.dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \sin nx.dx$$

$$= \frac{2}{\pi}\left[\left(\pi x - x^2\right)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2}{n^3}(1 - \cos n\pi)\right]$$

$$= \frac{4}{n\pi^3}\left(1 - (-1)^n\right)$$

$$b_n = \begin{cases} 0, \ when \ \ n \ \ is \ \ even \\ \dfrac{8}{\pi n^3}, \ when \ n \ is \ odd \end{cases}$$

Hence

$$x(\pi - x) = \sum_{n=1,3,5...} \frac{8}{\pi n^3} \sin nx \ \ (or)$$

$$x(\pi - x) = \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + ----\right) \rightarrow (1)$$

Deduction:-

Putting $x = \frac{\pi}{2}$ in (1), we get

$$\frac{\pi}{2}\left(x - \frac{\pi}{2}\right) = \frac{8}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3^3}\sin\frac{3\pi}{2} + \frac{1}{5^3} sin\frac{5\pi}{2} + --\right)$$

$$\frac{\pi^2}{4} = \frac{8}{\pi}\left[1 + \frac{1}{3^3}\sin\left(\pi + \frac{\pi}{2}\right) + \frac{1}{5^3}\sin\left(2\pi + \frac{\pi}{2}\right) + \frac{1}{7^3}\sin\left(3\pi + \frac{\pi}{2}\right) + ---\right]$$

$$(or)\frac{\pi^2}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + ---$$

3) Find the half- range sine series for the function $f(x) = \dfrac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} \ in \ \ (0, \pi)$

Ans. Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$then\ b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx.dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}}.\sin nx.dx$$

$$= \frac{2}{\pi\left(e^{a\pi}-e^{-a\pi}\right)}\left[\int_0^{\pi} e^{ax}\sin nx.dx - \int_0^{\pi} e^{-ax}\sin nx.dx\right]$$

$$= \frac{2}{\pi\left(e^{a\pi}-e^{-a\pi}\right)}\left[\left[\frac{e^{ax}}{a^2+n^2}\left(a\sin nx - n\cos nx\right)\right]_0^{\pi} - \left[\frac{e^{-ax}}{a^2+b^2}\left(-a\sin nx - n\cos nx\right)\right]_0^{\pi}\right]$$

$$= \frac{2}{\pi\left(e^{a\pi}-e^{-a\pi}\right)}\left[\frac{-e^{a\pi}}{a^2+n^2}n(-1)^n + \frac{n}{a^2+b^2} + \frac{e^{-a\pi}}{a^2+b^2}n(-1)^n - \frac{n}{a^2+b^2}\right]$$

$$= \frac{2n(-1)^n}{\pi\left(e^{a\pi}-e^{-a\pi}\right)}\left[\frac{e^{-a\pi}-e^{a\pi}}{n^2+a^2}\right]$$

$$= \frac{2n(-1)^{n+1}}{\pi\left(n^2+a^2\right)}$$

Substituting (2) in (1), we get

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{a^2+n^2}\sin nx$$

$$= \frac{2}{\pi}\left[\frac{\sin nx}{a^2+1^2} - \frac{2\sin 2x}{a^2+2^2} + \frac{3\sin 3x}{a^2+3^2} - - - - -\right]$$

Fourier series of $f(x)$ defined in $[c_1 c+2l]$

It can be seen that role played by the functions

$1, \cos x, \cos 2x, \cos 3x, ...\sin x, \sin 2x.........$

In expanding a function $f(x)$ defined in $[c_1 c+2\pi]$ as a Fourier series, will be played by

$1, \cos\left(\frac{\pi x}{e}\right), \cos\left(\frac{2\pi x}{e}\right), \cos\left(\frac{3\pi x}{e}\right), ...$

$\sin\left(\frac{\pi x}{e}\right), \sin\left(\frac{2\pi x}{e}\right), \sin\left(\frac{3\pi x}{e}\right), ...$

In expanding a function $f(x)$ defined in $[c, c+2I]$ it can be verified directly that, when m, n are integers

$$\int_c^{c+2l} \sin\left(\frac{m\pi x}{l}\right).\cos\left(\frac{2\pi x}{l}\right)dx = 0$$

$$\int_c^{c+2l} \sin\left(\frac{m\pi x}{l}\right).\sin\left(\frac{n\pi x}{l}\right)dx = \begin{cases} 0 & if \;\; m \neq n \\ l & if \;\; m = n \neq 0 \\ 2l & if \;\; m = 2n = 0 \end{cases}$$

$$\int_c^{c+2l} \cos\left(\frac{m\pi x}{l}\right).\cos\left(\frac{n\pi x}{l}\right)dx = \begin{cases} 0 & if \;\; m \neq n \\ l & if \;\; m = n \neq 0 \\ 2l & if \;\; m = n = 0 \end{cases}$$

Fourier series of $f(x)$ defined in $[0,2l]$:-

Let $f(x)$ be defined in $[0,2l]$ and be periodic with period $2l$. Its Fourier series expansion is defined as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right] \to (1)$$

$$where\ a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx\ and \to (2)$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx \to (3)$$

Fourier series of $f(x)$ defined in $[-l,l]$:-

Let $f(x)$ be defined in $[-l,l]$ and be periodic with period $2l$. Its Fourier series expansion is defined as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$where\ a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Fourier series for even and odd functions in $[-l,l]$:-

Let $f(x)$ be defined in $[-l,l]$. If $f(x)$ is even $f(x)\cos\frac{n\pi x}{l}$ is also even

$$\therefore a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx \quad and \quad f(x)\sin\frac{n\pi x}{l}\,is\ odd$$

$$\therefore b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = 0 \forall n$$

Hence if $f(x)$ is defined in $[-l,l]$ and is even its Fourier series expansion is given by

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

$$where\ a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

If $f(x)$ is defined in $[-l,l]$and its odd its Fourier series expansion is given by

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\quad where\ b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

Note:- In the above discussion if we put $2l=2\pi, l=\pi$ we get the discussion regarding the intervals $[0,2\pi]$ and $[-\pi,\pi]$ as special cases

Examples:-

1. Express $f(x)=x^2$ as a Fourier series in $[-l,l]$

Sol $f(-x)=f(-x)^2=x^2=f(x)$

Therefore $f(x)$ is an even function

Hence the Fourier series of $f(x)$ in $[-l,l]$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

$$where\ a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$hence\ a_0=\frac{2}{l}\int_0^l x^2dx=\frac{2}{l}\left(\frac{x^3}{3}\right)_0^l=\frac{2l}{3}$$

$$also\ a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\int_0^l x^2\cos\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\left[x^2\left[\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right]-2x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)+2\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$=\frac{2}{l}\left[2x\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]_0^l$$

Since the first and last terms vanish at both upper and lower limits

$$\therefore a_n=\frac{2}{l}\left[2l\frac{\cos n\pi}{n^2\pi^2/l^2}\right]=\frac{4l^2\cos n\pi}{n^2\pi^2}$$

$$=\frac{(-1)^n 4l^2}{n^2\pi^2}$$

Substituting these values in (1), we get

$$x^2 = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4l^2}{n^2\pi^2} \cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2} \left[\frac{\cos(\pi x/l)}{1^2} - \frac{\cos(2\pi x/l)}{2^2} + \frac{\cos(3\pi x/l)}{3^2} - - - - - \right]$$

2. Find a Fourier series with period 3 to represent $f(x) = x + x^2 \;\; in \; (0,3)$

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \right) \rightarrow (1)$

Here $2l = 3, \qquad l = 3/2$

Hence (1) becomes

$$f(x) = x + x^2$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2n\pi x}{3} + b_n \sin\frac{2n\pi x}{3} \right) \rightarrow (2)$$

Where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$

$$= \frac{2}{3}\int_0^3 \left(x + x^2\right) dx = \frac{2}{3}\left[\frac{x^2}{2} + \frac{x^3}{3} \right]_0^3 = 9$$

and $a_n = \frac{1}{l}\int_0^2 f(x) \cos\left(\frac{n\pi x}{l}\right) dx$

$$= \frac{2}{3}\int_0^3 \left(x + x^2\right) \cos\left(\frac{2n\pi x}{3}\right) dx$$

Integrating by parts, we obtain

$$a_n = \frac{2}{3}\left[\frac{3}{4n^2\pi^2 - 4n^2\pi^2} \right] = \frac{2}{3}\left(\frac{54}{9n^2\pi^2} \right) = \frac{9}{n^2\pi^2}$$

Finally $b_n = \frac{1}{l}\int_0^{2l} f(x) \sin\frac{n\pi x}{l}\,dx$

$$= \frac{2}{3}\int_0^3 \left(x + x^2\right) \sin\left(\frac{2n\pi x}{3}\right) dx$$

$$= \frac{-12}{n\pi}$$

Substituting the values of a’s and b’s in (2) we get

$$x + x^2 = \frac{9}{2} + \frac{9}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{2n\pi x}{3}\right) - \frac{12}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{2n\pi x}{3}\right)$$

Half- Range Expansion of $f(x)\ in\ [0,l]$:-

Some times we will be interested in finding the expansion of $f(x)$ defined in $[0,l]$ in terms of sines only (or) in terms of cosines only. Suppose we want the expansion of $f(x)$ in terms of sine series only. Define $f_1(x)=f(x)$ in $[0,l]$ and $f_1(x)=-f_1(x)\forall n$ with $f_1[2l+x]=f_1(x)$, $f_1(x)$ is an odd function in $[-l,l]$. Hence its Fourier series expansion is given by

$$f_1(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}dx$$

$$where\ b_n=\frac{2}{l}\int_0^1 f_1(x)dx$$

The above expansion is valid for $x\ in\ [-l,l]$ in particular for $x\ in\ [0,l]$,

$f_1(x)=f(x)$ and $f_1(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}dx$ where $b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$

This expansion in (3) is called the half- range sine series expansion of $f(x)$ in $[0,l]$. If we want the half – range expansion of $f(x)$ in $[0,l]$, only in terms of cosines, define $f_1(x)=f(x)$ in $[0,l]$ and $f_1(-x)=f_1(x)$ for all x with

$$f_1(x+2l)=f_1(x).$$

Then $f_1(x)$ is even in $[-l,l]$ and hence its Fourier series expansion is given

By

$$f_1(x)=\frac{a_0}{2}+\sum a_n\cos\frac{n\pi x}{l}$$

$$where\ a_n=\frac{2}{l}\int_0^l f_1(x)\cos\frac{n\pi x}{l}dx$$

The expansion is valid in $[-l,l]$ and hence in particular on $[0,l]$, $f_1(x)=f(x)$ hence in $[0,l]$

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$$

Where $a_n=\frac{2}{l}\int_0^1 f(x)\cos\frac{n\pi x}{l}dx$

1. The half range sine series expansion of $f(x)=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$ in $(0,2)$ is given by

   Where $b_n=\frac{2}{l}\int_0^1 f(x)\sin\frac{n\pi x}{l}dx$

2. The half range cosine series expansion of $f(x)\ in\ [0,l]$ is given by

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$$

$$where\ b_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

<u>Examples:-</u>

1. Find the half- range sine series of $f(x)=1\ in\ [0,l]$

Ans. The Fourier sine series of $f(x)$ $in\,[0,l]$ is given by $f(x)=1=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$$here\ b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\int_0^l 1.\sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right)_0^l$$

$$=\frac{2}{n\pi}\left[-\cos\frac{n\pi x}{l}\right]_0^l$$

$$=\frac{2}{n\pi}(-\cos n\pi+1)$$

$$=\frac{2}{n\pi}\left[(-1)^{n+1}+1\right]$$

$\therefore b_n=0$ when n is even

$=\frac{4}{n\pi}$, when n is odd

Hence the required Fourier series is $f(x)=\sum_{n=1,3,5---}^{\infty}\frac{4}{n\pi}\sin\frac{n\pi x}{l}$

2. Find the half – range cosine series expansion of $f(x)=\sin\left(\frac{\pi x}{l}\right)$ in the range $0<x<l$

Sol

$$f(xa_0)=\sin\left(\frac{\pi x}{l}\right)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

$$where\ a_0=\frac{2}{l}\int_0^l f(x)dx=\frac{2}{l}\int_0^l\sin\frac{\pi x}{l}dx$$

$$=\frac{2}{l}\left[\frac{-\cos\pi x/l}{\pi/l}\right]_0^l$$

$$=\frac{2}{l}(\cos\pi-1)=\frac{4}{\pi}\ and$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\int_0^l\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$=\frac{1}{l}\int_0^l\left[\frac{\sin(n+1)\pi x}{l}-\frac{\sin(n-1)\pi x}{l}\right]dx$$

$$=\frac{1}{l}\left[-\frac{\frac{\cos(n+1)\pi x}{l}}{(n+1)\pi/l}+\frac{\cos(n-1)\pi x/l}{(n-1)\pi/l}\right]_0^l$$

$$=\frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]$$

When n is odd

$$a_n=\frac{1}{\pi}\left[\frac{-1}{n+1}+\frac{1}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]=0$$

When n is even

$$a_n=\frac{1}{\pi}\left[\frac{1}{n+1}-\frac{1}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$=\frac{-4}{\pi(n+1)(n-1)}$$

$$\therefore \sin\left(\frac{\pi x}{l}\right)=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos(2\pi x/l)}{1.3}+\frac{\cos(4\pi x/l)}{3.5}+----\right]$$

Fourier Transforms

Fourier Transforms are widely used to solve Partial Differential Equations and in various boundary value problems of Engineering such as Vibration of Strings, Conduction of heat, Oscillation of an elastic beam, Transmission lines etc.

Integral Transforms:

The Integral transform of a function f(x) is defined as

I{f(x)} = $\bar{f}(s) = \int_{x=x_1}^{x_2} f(x)K(s,x)dx$

Where K(s,x) is a known function of s & x, called the 'Kernel' of the transform.
The function f(x) is called the Inverse transform of $\bar{f}(s)$

1.**Laplace Transform**: When K(s,x) = e^{-sx}

L{f(x)} = $\bar{f}(s) = \int_0^\infty f(x)e^{-sx}dx$

2.**Fourier Transform**: When K(s,x) = e^{isx}

F{f(x)} = $\bar{f}(s) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)e^{isx}dx$

3.**Fourier Sine Transform**: When K(s,x)=Sinsx

F_s\{f(x)\} = $\bar{f}(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)sinsx\ dx$

4. **Fourier Cosine Transform**: When K(s,x)=Cossx

F_c\{f(x)\} = $\bar{f}(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)cossx\ dx$

5.**Mellin Transform**: When K(s,x)= x^{s-1}

M$(s) = \int_0^\infty f(x)x^{s-1}dx$

6.**Hankel Transform**: When K(s,x) = x$J_n(sx)$

H(s) = $\bar{f}(s) = \int_0^\infty f(x)xJ_n(sx)dx$

Where $J_n(sx)$ is a Bessel function.

Fourier Integral Theorem:- If f(x) satisfies Dirichlet's conditions for expansion of Fourier series in (-c,c) and $\int_{-\infty}^{\infty}|f(x)|$ converges, then

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dt\ d\lambda$$

Which is known as Fourier Integral of f(x)

Proof: Since f(x) satisfies Dirichlet's conditions in (-c,c), Fourier series of f(x) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{c} + b_n\sin\frac{n\pi x}{c}) \qquad \ldots\ldots\ldots\ldots\ldots.(1)$$

Where $a_0 = \frac{1}{c}\int_{-c}^{c} f(t)dt$, $a_n = \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}dt$, $b_n = \frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}dt$

Substitute the values of a_0, a_n and b_n in (1) , we get

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)dt + \frac{1}{c}\sum_{n=1}^{\infty}\int_{-c}^{c} f(t)\cos\frac{n\pi(t-x)}{c}\ dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

Since $\int_{-\infty}^{\infty}|f(x)|\ dx$ converges i.e., f(x) is absolutely integrable on x-axis,

The first term on R.H.S of (2) approaches to '0' as $c \to \infty$

Since $\left|\frac{1}{2c}\int_{-\infty}^{\infty} f(t)dt\right| \le \frac{1}{2c}\int_{-\infty}^{\infty}|f(t)|\ dt$

The second term on R.H.S of (2) tends to

$$Lt_{c\to\infty}\frac{1}{c}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\frac{n\pi(t-x)}{c}\ dt = Lt_{\frac{\pi}{c}\to 0}\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\frac{n\pi(t-x)}{c}\ dt$$

Let $\frac{\pi}{c} = \delta\lambda$ $so\ that\ \delta\lambda \to 0\ as\ c \to \infty$

$$f(x) = Lt_{\delta\lambda\to 0}\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f(t)\cos n(t-x)\delta\lambda\ dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$$

This is of the form $Lt_{\delta\lambda\to 0}\sum_{n=1}^{\infty}F(n\delta\lambda)$ $i.e.,$ $\int_0^{\infty}F(\lambda)\,d\lambda$

Thus $as\ c \to \infty$, (3) becomes

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dt\ d\lambda$$

Which is known as Fourier Integral of f(x)

Fourier Sine & Cosine Integrals:-

From Fourier Integral theorem

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dt\ d\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

w.k.t cosλ(t-x) = cosλt cosλx + sinλt sinλx

Sub. This value in eq(1), we get

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\cos\lambda x\int_{-\infty}^{\infty} f(t)\cos\lambda t\ dt\ d\lambda + \frac{1}{\pi}\int_0^{\infty}\sin\lambda x\int_{-\infty}^{\infty} f(t)\sin\lambda t\ dt\ d\lambda \ \ldots\ldots.(2)$$

when f(t) is odd function, then f(t)cosλt is an odd function while f(t)sinλt is an even function.
then eq(2) becomes

f(x) = $\frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda t \, dt \, d\lambda$

This is called "Fourier sine Integral"

when f(t) is even function then f(t) cosλt is an even function, while f(t)sinλt is an odd function
then eq(2) becomes

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda t \, dt \, d\lambda$$

This is called "Fourier cosine Integral"

Complex form of Fourier Integral:-

From Fourier Integral theorem

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\lambda(t-x)dt \, d\lambda \quad \text{…………..(1)}$$

$$= \frac{1}{\pi}\int_{-\infty}^\infty f(t)dt \, \{\int_0^\infty \cos\lambda(t-x)d\lambda\}$$

since cos λ(t-x) is an even function

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\cos\lambda(t-x)dt \, d\lambda \quad \text{………………..(2)}$$

w.k.t sinλ(t-x) is an odd function,

$$\int_{-\infty}^\infty \sin\lambda(t-x) \, d\lambda = 0$$

$$\frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\sin\lambda(t-x)dt \, d\lambda = 0 \quad \text{……………………(3)}$$

multiply (3) by i and add it to (2), then

f(x) = $\frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)[\cos\lambda(t-x) + i\sin\lambda(t-x)]dt \, d\lambda$

$= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \, e^{i\lambda(t-x)} dt \, d\lambda$

which is known as 'Complex form of Fourier Integral'.

Problems:

1.Express the function f(x) = $\begin{cases} 1; & |x| \le 1 \\ 0; & |x| > 1 \end{cases}$ as a Fourier integral and hence evaluate

$$\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda} d\lambda$$

sol: The Fourier Integral of f(x) is given by

f(x) = $\frac{1}{\pi}\int_{\lambda=0}^\infty d\lambda \int_{t=-\infty}^\infty f(t)\cos\lambda(t-x)dt$ …………..(1)

given that f(t) = $\begin{cases} 0; & -\infty < t < -1 \\ 1; & -1 < t < 1 \\ 0; & 1 < t < \infty \end{cases}$

f(x) = $\frac{1}{\pi}\int_{\lambda=0}^\infty d\lambda \int_{t=-1}^1 \cos\lambda(t-x)dt$

$= \frac{1}{\pi}\int_{\lambda=0}^{\infty}(\frac{sin\lambda(t-x)}{\lambda})_{t=-1}^{1} d\lambda$

$= \frac{1}{\pi}\int_{\lambda=0}^{\infty} \frac{1}{\lambda}\,(sin\lambda(1-x) - sin\lambda(-1-x))d\lambda$

$= \frac{1}{\pi}\int_{\lambda=0}^{\infty} \frac{1}{\lambda}\,(sin\lambda(1-x) + sin\lambda(1+x))d\lambda$

$= \frac{2}{\pi}\int_{\lambda=0}^{\infty} \frac{sin\lambda cos\lambda x}{\lambda} d\lambda$ ……………………(2)

which is fourier integral of f(x)

from (2), $\frac{2}{\pi}\int_{\lambda=0}^{\infty} \frac{sin\lambda cos\lambda x}{\lambda} d\lambda$ = f(x)

$\int_{\lambda=0}^{\infty} \frac{sin\lambda cos\lambda x}{\lambda} d\lambda = \frac{\pi}{2} f(x)$

given f(x) = $\begin{cases} 1; & |x| \leq 1 \\ 0; & |x| > 1 \end{cases}$

$\int_{\lambda=0}^{\infty} \frac{sin\lambda cos\lambda x}{\lambda} d\lambda = \begin{cases} \frac{\pi}{2}; & |x| < 1 \\ 0; & |x| > 1 \end{cases}$

at $|x| = 1$ i.e., when x=±1

f(x) is discontinuous & the integral has the value $\frac{1}{2}(\frac{\pi}{2} + 0) = \frac{\pi}{4}$

$\int_{\lambda=0}^{\infty} \frac{sin\lambda cos\lambda x}{\lambda} d\lambda = \frac{\pi}{4} \ at\ |x| = 1$

2.Find Fourier Sine Integral representation of f(x) = $\begin{cases} 0, \infty < x < 1 \\ x, -1 < x < 0 \\ 0, \quad x > 0 \end{cases}$

sol: Fourier Sine integral of f(x) is given by

f(x) = $\frac{2}{\pi}\int_0^{\infty} sin\lambda x \int_0^{\infty} f(t) sin\lambda t\ dt\ d\lambda$ ……………..(1)

given that f(t) = $\begin{cases} t, & -1 < t < 0 \\ 0, & else\ where \end{cases}$

f(x) = $\frac{2}{\pi}\int_0^{\infty} sin\lambda x\ \{\int_{-1}^{0} tsin\lambda t\ dt\}\ d\lambda$

$= \frac{2}{\pi}\int_0^{\infty} sin\lambda x \left(\frac{-tcos\lambda t}{\lambda} + \frac{sin\lambda t}{\lambda^2}\right)_{t=-1}^{0} d\lambda$

f(x) = $\frac{2}{\pi}\int_0^{\infty} sin\lambda x\ (\frac{sin\lambda}{\lambda^2} - \frac{cos\lambda}{\lambda})d\lambda$

Using Fourier Integral, show that $\int_0^\infty \frac{1-\cos \pi s}{s} \sin sx \, ds = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$

Solution:

Let $f(x) = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$ ....(1)

We know that, Fourier sine Integral is given by

$$f(x) = \frac{2}{\pi} \int_{t=0}^{\infty} \sin sx \int_{s=0}^{\infty} f(t) \sin st \, dt \, ds \quad ...(2)$$

Substituting (1) in (2), we have

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin sx \left[\int_0^\pi f(t) \sin st \, dt \, ds \right]$$

$$= \frac{2}{\pi} \int_0^\infty \sin sx \left[\int_0^\pi \frac{\pi}{2} \sin st \, dt \right] ds$$

$$\Rightarrow f(x) = \frac{\pi}{2} \cdot \frac{2}{\pi} \int_0^\infty \sin sx \left[\frac{-(\cos st)}{s} \right]_0^\pi ds$$

$$\Rightarrow f(x) = \int_0^\infty \left[\frac{1-(\cos s\pi)}{s} \right] \sin sx \, ds$$

$$\Rightarrow \int_0^\infty \frac{1-(\cos s\pi)}{s} \sin sx \, ds = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$$

Using Fourier Integral, show that $e^{-x} \cos x = \frac{2}{s} \int_0^\infty \frac{s^2+2}{s^4+4} \cos sx \, ds$

Given $e^{-x} \cos x = \frac{2}{s} \int_0^\infty \frac{s^2+2}{s^4+4} \cos sx \, ds$

Since the integrand contains cosine terms, by Fourier cosine integral we have,

$$f(x) = \frac{2}{p} \int_0^\infty \int_0^\infty \cos px \, f(t) \cos pt \, dt \, dp$$

Replacing 'p' by 's' in the above equation we have,

$$f(x) = \frac{2}{p} \int_0^\infty \int_0^\infty f(t) \cos st \cos sx \, dt \, ds \quad \text{let } f(x) = e^{-x} \cos x$$

$$\therefore e^{-x} \cos x = \frac{2}{p} \int_0^\infty \int_0^\infty e^{-t} \cos t \cos st \cos sx \, dt \, ds$$

$$= \frac{1}{p} \int_0^\infty \left[\int_0^\infty e^{-t} (2\cos t \cos st) \, dt \right] \cos sx \, ds$$

$$= \frac{1}{p}\int_0^\infty\left[\int_0^\infty e^{-t}(\cos(s+1)t+\cos(s-1)t)dt\right]\cos sx\, ds \qquad \left[\therefore \int_0^\infty e^{-ax}\cos bx\, dx = \frac{a}{a^2+b^2}\right]$$

$$\therefore e^{-x}\cos x = \frac{1}{p}\int_0^\infty\left[\frac{1}{(s+1)^2+1)} + \frac{1}{(s-1)^2+1}\right]\cos sx\, ds$$

$$= \frac{1}{p}\int_0^\infty\left[\frac{2(s+2)\cos sx}{((s^2+2)+2s)((s^2+2)-2s)}\right]\cos sx\, ds$$

$$= \frac{2}{p}\int_0^\infty\left[\frac{(s^2+2)\cos sx}{(s^2+2)^2-(2s)^2}\right]ds$$

$$\therefore e^{-x}\cos x = \frac{2}{p}\int_0^\infty \frac{s^2+2}{s^4+4}\cos sx\, ds$$

Fourier Transforms:-

Complex form of Fourier Integral of f(x) is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\lambda(t-x)}\, dt\, d\lambda$$

replace λ by s

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}ds \,.\, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ist}dt$$

If we define $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ist}dt$

then $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$

F(s) is called Fourier Transform (F.T) of f(x) and f(x) is called inverse Fourier transform of F(s)

Fourier Sine & Cosine transforms:-

The Fourier sine integral of f(x) is defined as

$$f(x) = \frac{2}{\pi}\int_0^\infty sinsx \int_0^\infty f(x) sinsx\, dx\, ds$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty sinsx\, ds. \sqrt{\frac{2}{\pi}}\int_0^\infty f(x) sinsx\, dx$$

If we define $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

then f(x) = $\sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$

here $\boldsymbol{F_s(s)}$ is called **Fourier sine transform** of **f(x)** and **f(x)** is called **Inverse Fourier sine transform** of $\boldsymbol{F_s(s)}$

similarly, Fourier cosine integral of f(x) is

f(x) = $\frac{2}{\pi} \int_0^\infty \cos sx \int_0^\infty f(x) \cos sx \, dx \, ds$

if we define $F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

then f(x) = $\sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$

here $\boldsymbol{F_c(s)}$ is called **Fourier cosine transform** of **f(x)** and **f(x)** is called **Inverse Fourier cosine transform** of $\boldsymbol{F_c(s)}$

NOTE: 1. Some authors define F.T as follows

i) F(s) = $\int_{-\infty}^{\infty} f(x) e^{isx} dx$ ii) f(x) = $\frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$

iii)F(s) = $\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx} dx$ iv) f(x) = $\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} F(s) e^{isx} ds$

2.Some authors define Fourier sine & cosine transforms as follows

i) $F_s(s) = \int_0^\infty f(x) \sin sx \, dx$ ii) f(x) = $\frac{2}{\pi} \int_0^\infty F_s(s) \sin sx \, ds$

iii) $F_c(s) = \int_0^\infty f(x) \cos sx \, dx$ iv) f(x) = $\frac{2}{\pi} \int_0^\infty F_c(s) \cos sx \, ds$

Properties of Fourier Transforms:-

1.**Linearity Property**:- If $F_1(s)$ and $F_2(s)$ be the Fourier transforms of $f_1(x)$ and $f_2(x)$ respectively then f{a$f_1(x) + b\, f_2(x)\} = a\, F_1(s) + b\, F_2(s)$, where a & b are constants

proof:- by definition of Fourier transform,

F{ a$f_1(x) + b\, f_2(x)\}$ = $\int_{-\infty}^{\infty} e^{isx} (af_1(x) + b\, f_2(x))\, dx$

= a$\int_{-\infty}^{\infty} e^{isx} f_1(x)\, dx$ + b$\int_{-\infty}^{\infty} e^{isx} f_2(x)\, dx$

= $a\, F_1(s) + b\, F_2(s)$

2.**Change of Scale Property**:- If F{f(x)} = F(s) then F{f(ax)} = $\frac{1}{a} F(\frac{s}{a})$

proof:- By definition of F.T,

$$F\{f(x)\} = F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx \text{ ....(1)}$$

$$F\{f(ax)\} = \int_{-\infty}^{\infty} e^{isx} f(ax)\, dx$$

put ax = t

then a dx = dt

$$F\{f(ax)\} = \int_{-\infty}^{\infty} e^{is(\frac{t}{a})} f(t)\, dt/a$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} e^{i(\frac{s}{a})t} f(t)\, dt$$

$$= \frac{1}{a}\, F(\frac{s}{a}) \qquad \text{by(1)}$$

3.**<u>Shifting Property</u>**:- If F{f(x)} = F(s) then F{f(x-a)} = $e^{isa} F(s)$

Proof:- By definition of F.T,

$$F\{f(x)\} = F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx \text{ ....(1)}$$

$$F\{f(x-a)\} = \int_{-\infty}^{\infty} e^{isx} f(x-a)\, dx$$

put x-a = t

then dx = dt

$$F\{f(x-a)\} = \int_{-\infty}^{\infty} e^{is(t+a)} f(t)\, dt$$

$$= e^{isa} \int_{-\infty}^{\infty} e^{ist} f(t)\, dt$$

$$= e^{isa}\, F(s) \qquad \text{by(1)}$$

4.**<u>Modulation Property</u>**:- If F{f(x)} = F(s) then F{f(x)cosax} = ½ {F(s+a)+F(s-a)}

Proof:- By definition of F.T,

$$F\{f(x)\} = F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx \text{ ....(1)}$$

$$F\{f(x)\cos ax\} = \int_{-\infty}^{\infty} e^{isx} f(x)\cos ax\, dx$$

$$= \int_{-\infty}^{\infty} e^{isx} f(x)(\frac{e^{iax}+e^{-iax}}{2})\, dx$$

$$= \frac{1}{2}\{ \int_{-\infty}^{\infty} e^{i(s+a)} f(x)dx + \int_{-\infty}^{\infty} e^{i(s-a)} f(x)dx\}$$

$$= \frac{1}{2} \{F(s+a) + F(s-a)\}$$

5.**<u>Convolution Property</u>**:- The convolution of two functions f(t) and g(t) in (-∞,∞) is defined as f(t) * g(t) = $\int_{-\infty}^{\infty} f(t).g(x-t)\, dt$

<u>Theorem:</u>- If F{f(t)}=$F_1(s)$ and F{g(t)}= $F_2(s)$ then F{f(t)*g(t)} = $F_1(s).F_2(s)$

Proof:- By definition of F.T we have

$F\{f(t)*g(t)\} = \int_{-\infty}^{\infty} \big(f(t) * g(t)\big) e^{ist} dt$

$= \int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} f(u) g(t-u) du\} e^{ist} dt$

$= \int_{-\infty}^{\infty} f(u) e^{isu} \{\int_{-\infty}^{\infty} g(t-u) e^{is(t-u)} d(t-u)\} du$

on changing the order of integration,

$= \int_{-\infty}^{\infty} f(u) e^{isu} . F_2(s) du$

$= \{\int_{-\infty}^{\infty} f(u) e^{isu} du\} . F_2(s)$

$= F_1(s) . F_2(s)$

$F\{f(t)*g(t)\} = F_1(s) . F_2(s)$

6.If F{f(x)} = F(s) then F{f(-x)}= F(-s)

Proof: By definition, $F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{isx} dx$ …………..(1)

$F\{f(-x)\} = \int_{-\infty}^{\infty} f(-x) e^{isx} dx$

put –x = t then dx = -dt

as x⟶∞, t⟶-∞ and as x⟶-∞ , t⟶∞

$F\{f(x)\} = \int_{\infty}^{-\infty} f(t) e^{-ist} (-dt)$

$= \int_{-\infty}^{\infty} f(t) e^{-ist} dt$

$= \int_{-\infty}^{\infty} f(t) e^{i(-s)t} dt$

= F(-s) (by (1))

7.$\overline{F\{f(x)\}} = \overline{F(-s)}$

Proof: By definition, $F\{f(x)\} = F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$ …………..(1)

$F(-s) = \int_{-\infty}^{\infty} f(x) e^{-isx} dx$

taking complex conjugate on both sides

$\overline{F(-s)} = \int_{-\infty}^{\infty} \overline{f(x)} e^{isx} dx$

$= \overline{F\{f(x)\}}$

8. $\overline{F\{f(-x)\}} = \overline{F(s)}$

Proof: By definition, $F\{f(x)\} = F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$

take complex conjugate on both sides

$\overline{F(s)} = \int_{-\infty}^{\infty} \overline{f(x)} e^{-isx} dx$

put x=-z then dx = -dz

$\overline{F(s)} = \int_{\infty}^{-\infty} \overline{f(-z)} e^{isz} (-dz)$

$= \int_{-\infty}^{\infty} \overline{f(-z)} e^{isz} dz$

$= F\{\overline{f(-x)}\}$

$$\overline{F\{f(-x)\}} = \overline{F(s)}$$

9. $F_c\{xf(x)\} = \frac{d}{ds} F_s\{f(x)\}$

Proof: By definition of Fourier sine transform

$$F_s\{f(x)\} = \int_0^\infty f(x) sinsx\, dx$$

$$\frac{d}{ds} F_s\{f(x)\} = \frac{d}{ds}\{ \int_0^\infty f(x) sinsx\, dx \}$$

$$= \int_0^\infty \frac{d}{ds}\{ f(x) sinsx\} dx$$

$$= \int_0^\infty f(x).xcossx\, dx$$

$$= \int_0^\infty \{xf(x)\} cossx\, dx$$

$$= F_c\{xf(x)\}$$

Note: $F_s\{xf(x)\} = -\frac{d}{ds} F_c\{f(x)\}$

Problems:

1. Find the F.T of f(x) = $e^{-|x|}$

sol: Given f(x) = $e^{-|x|}$

$$= \begin{cases} e^{x}; x < 0 \\ e^{-x}; x > 0 \end{cases}$$

by definition, F{f(x)} = $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}}\{\int_{-\infty}^{0} f(x) e^{isx} dx + \int_0^\infty f(x) e^{isx} dx\}$$

$$= \frac{1}{\sqrt{2\pi}}\{\int_{-\infty}^{0} e^{(1+is)x} dx + \int_0^\infty e^{(-1+is)x} dx\}$$

$$= \frac{1}{\sqrt{2\pi}} \{ \left(\frac{e^{(1+is)x}}{1+is}\right)_{-\infty}^{0} + \left(\frac{-e^{-(1-is)x}}{1-is}\right)_0^\infty \}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1+is} + \frac{1}{1-is}\right)$$

$$= \sqrt{\frac{2}{\pi}} . \frac{1}{1+s^2}$$

2. S.T the Fourier Sine transform of f(x) = $\begin{cases} x, 0 < x \leq 1 \\ 2-x, 1 < x < 2 \\ 0, x > 2 \end{cases}$ **is** $\frac{2sins(1-coss)}{s^2}$

sol: By definition,

$$F_s\{f(x)\} = \int_0^\infty f(x) sinsx\, dx$$

$$= \int_0^1 f(x) sinsx\, dx + \int_1^2 f(x) sinsx\, dx + \int_2^\infty f(x) sinsx\, dx$$

$$= \int_0^1 x sinsx\, dx + \int_1^2 (2-x) sinsx\, dx$$

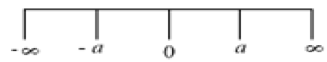
$$= [x.\left(\frac{-\cos sx}{s}\right) - \left(\frac{-\sin sx}{s^2}\right)]_0^1 + [(2-x).\left(\frac{-\cos sx}{s}\right) - (-1)\left(\frac{-\sin sx}{s^2}\right)]_1^2$$

$$= \frac{-\cos s}{s} + \frac{\sin s}{s^2} - \frac{\sin 2s}{s^2} + \frac{\cos s}{s} + \frac{\sin s}{s^2}$$

$$= \frac{2\sin s\,(1-\cos s)}{s^2}$$

Find the Fourier Transform of $F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ and hence evaluate (i) $\int_{-\infty}^{\infty} \frac{\sin as \ \cos sx}{s} ds$ (ii) $\int_0^{\infty} \frac{\sin s}{s} ds$

Given $F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases} \Rightarrow F(x) = \begin{cases} 1, & -a < x < a \\ 0, & |x| > a \end{cases}$



By definition of Fourier Transform, we have,

$$F\{F(x)\} = f(s) = \int_{x=-\infty}^{\infty} e^{isx} F(x)\, dx$$

$$\Rightarrow \int_{x=-\infty}^{-a} e^{isx} F(x)\, dx + \int_{x=-a}^{+a} e^{isx} F(x)\, dx + \int_{x=a}^{\infty} e^{isx} F(x)\, dx$$

$$\Rightarrow \int_{-\infty}^{x=-a} e^{isx} F(x)\, dx + \int_{x=-a}^{+a} e^{isx} F(x)\, dx + \int_{x=-a}^{-\infty} e^{isx} F(x)\, dx$$

$$\Rightarrow \int_{x=-\infty}^{-a} e^{isx} .0\, dx + \int_{x=-a}^{a} e^{isx} .\ 1\, dx + \int_{x=a}^{\infty} e^{isx} .\ 0\, dx$$

$$\Rightarrow \int_{x=-a}^{a} e^{isx}\, dx \Rightarrow \left[\frac{e^{isx}}{is}\right]_{x=-a}^{a} \Rightarrow \frac{1}{is}\ [e^{isa} - e^{-isa}]$$

$$\therefore\ F\{F(x)\} = \frac{2}{s}\sin sa = f(s) \quad \left[\because \frac{e^{isx} - e^{-isx}}{2i} = \sin sx\right]$$

$$F^{-1}\{F(x)\} = f(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, e^{-isx}\, ds$$

$$\left[\because \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin sa}{s} e^{-isx}\, ds = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}\right]$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin sa}{s} (\cos sx - i\sin sx)\, ds = F(x) \qquad [\because e^{-isx} = \cos x - i\sin sx]$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin sa.\cos sx}{s} - \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin sa\sin sx}{s} = F(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin sa\cos sx\, ds}{s} - 0 = F(x)$$

$$\left[\because 2^{nd} \text{ Integral is an odd function } \& \int_{-a}^{a} F(x)\, dx = 0 \text{ if } F(x) \text{ is odd}\right]$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin sa\cos sx}{s}\,ds = \begin{cases}1, & |x| < a\\ 0, & |x| > a\end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\sin sa\cos sx}{s}\,ds = \begin{cases}\frac{2\pi}{2} & |x| < a\\ 0, & |x| > a\end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\sin sa\cos sx}{s}\,ds = \begin{cases}\pi, & |x| < a\\ 0, & |x| > a\end{cases} \longrightarrow (1)$$

put $x = 0$ & $a = 1$ in (1) we get $\int_{-\infty}^{\infty}\frac{\sin s\cos 0}{s}\,ds = \pi$

$$\Rightarrow 2\int_{0}^{\infty}\frac{\sin s}{s}\,ds = \pi$$

$\left[\because \frac{\sin s}{s}\right.$ is an even function & $\int_{-a}^{a}F(x)\,dx$ where $F(x)$ is even$\left.\right]$

$$\Rightarrow \int_{0}^{\infty}\frac{\sin s}{s}\,ds = \frac{\pi}{2}$$

Examples on Infinite Fourier Transform

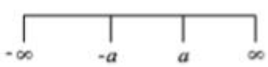
Find the Fourier Transform of $F(x) = \begin{cases}1-|x|^2, & |x| \le 1\\ 0, & |x| > 1\end{cases}$ Hence evaluate

(i) $\int_{0}^{\infty}\frac{s\cos s - \sin s}{s^3}\cos\frac{s}{2}\,ds$

(ii) $\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\,ds$

Given $F(x) = \begin{cases}1-|x|^2, & |x| \le 1 \text{ i.e., } -1 \le x \le 1\\ 0, & |x| > 1\end{cases}$ --- (1)

By definition of Fourier transforms form, we have,



$$F\{F(x)\} = f(s) = \int_{-\infty}^{\infty} e^{isx}\,F(x)\,dx$$

$$= \int_{-\infty}^{-1}F(x)\,e^{isx}\,dx + \int_{-1}^{1}F(x)\,e^{isx}\,dx + \int_{1}^{\infty}F(x)\,e^{isx}\,dx$$

Examples on Infinite Fourier Transform

Find the Fourier Transform of $F(x) = \begin{cases}1-|x|^2, & |x| \le 1\\ 0, & |x| > 1\end{cases}$ Hence evaluate

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(ii) $\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\,ds$

Given $F(x) = \begin{cases}1-|x|^2, & |x| \le 1 \text{ i.e., } -1 \le x \le 1\\ 0, & |x| > 1\end{cases}$ --- (1)

Examples on Infinite Fourier Transform

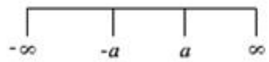
Find the Fourier Transform of $F(x) = \begin{cases} 1-|x|^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ Hence evaluate

(i) $\int_0^\infty \frac{s\cos s - \sin s}{s^3} \cos\frac{s}{2}\, ds$

(ii) $\int_{-\infty}^\infty \frac{s\cos s - \sin s}{s^3}\, ds$

Given $F(x) = \begin{cases} 1 - |x|^2, & |x| \le 1 \text{ i.e., } -1 \le x \le 1 \\ 0, & |x| > 1 \end{cases}$ --- (1)

By definition of Fourier transforms form, we have,



$$F\{F(x)\} = f(s) = \int_{-\infty}^{\infty} e^{isx} F(x)\, dx$$

$$= \int_{-\infty}^{-1} F(x) e^{isx} dx + \int_{-1}^{1} F(x) e^{isx} dx + \int_{1}^{\infty} F(x) e^{isx} dx$$

$$= -\int_{1}^{\infty} F(x) e^{isx} dx + \int_{-1}^{1} F(x) e^{isx} dx + \int_{1}^{\infty} F(x) e^{isx} dx$$

$$= +\int_{1}^{\infty} F(x) e^{isx} dx + \int_{-1}^{1} F(x) e^{isx} dx + \int_{1}^{\infty} F(x) e^{isx} dx = 0 + \int_{x=-1}^{+1} (1 - x^2)\, e^{isx}\, dx + 0$$

On integrating by parts we get, $\left[(1 - x^2)\frac{e^{isx}}{is}\right]_{x=-1}^{1} - \int_{-1}^{1} (-2x)\frac{e^{isx}}{is}\, dx$

$$\left[(1 - x^2)\frac{e^{isx}}{is} - \frac{(1-(1)^2)\, e^{-is}}{is}\right]^1 + \int_{-1}^{1} \frac{2}{is}\, x e^{isx}\, dx$$

Again integrating by parts we have,

$$\Rightarrow F\{F(x)\} = F(x) = \frac{2}{is}\left[\left(\frac{1.e^{is}}{is} - \frac{(-1)\, e^{-is}}{is}\right) - \left(\frac{e^{is}}{i^2 s^2}\right)_{-1}^{1}\right]$$

$$\Rightarrow F\{F(x)\} = F(x) = \frac{2}{is}\left[\left(\frac{e^{is} + e^{-is}}{is}\right) + \left(\frac{e^{is} - e^{-is}}{is}\right)\right]$$

$$\Rightarrow F\{F(x)\} = f(s) = \frac{2}{is}\left[\frac{2\cos s}{is} + \frac{1}{s^2}(2i\sin s)\right]$$

$$\left[\because \cos sx = \frac{e^{isx} + e^{-isx}}{2}, \quad \sin sx = \frac{e^{isx} - e^{-isx}}{2i}\right]$$

$$\Rightarrow F\{F(x)\} = f(s) = 4\left[\frac{\cos s}{s^2} + \frac{\sin s}{s^3}\right]$$

$$\Rightarrow F\{F(x)\} = f(s) = 4\left[\frac{-s\cos s}{s^3} + \frac{\sin s}{s^3}\right]$$

$$\Rightarrow F\{F(x)\} = f(s) = 4\left[\frac{-s\cos s + \sin s}{s^3}\right]$$

To evaluate $\int_0^\infty \frac{s\cos s - \sin s}{s^3} \cos\frac{s}{2}\, ds$

By inversion formula for complex infinite Fourier transform, we have,

$$F^{-1}\{f(s)\} = F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, e^{-isx}\, ds$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{s^3}(-s\cos s + \sin s)\, e^{-isx}\; ds = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \quad \text{By (2)}$$

put $x = \frac{1}{2}$ in (2) we get, $\Rightarrow -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{s^3}(s\cos s - \sin s)\, e^{-isx/2}\, ds = 1 - \frac{1}{4}$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{4}{s^3}(s\cos s - \sin s)\left(\cos\frac{s}{2} - i\sin\frac{s}{2}\right) ds = \frac{-3\pi}{8}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{(s\cos s - \sin s)}{s^3}\cos\frac{s}{2}\; ds - i\int_{-\infty}^{\infty}\frac{1}{s^3}(s\cos s - \sin s)\sin\frac{s}{2}\; ds = \frac{-3\pi}{8}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{(s\cos s - \sin s)}{s^3}\cos\frac{s}{2}\; ds = \frac{-3\pi}{8}\left[\frac{1}{s^3}(s\cos s - \sin s)\sin\frac{s}{2}\, ds = 0, \text{ since it is an odd function}\right]$$

$$\Rightarrow 2\int_0^{\infty} \frac{(s\cos s - \sin s)}{s^3}\cos\frac{s}{2}\; ds = \frac{-3\pi}{8}\left[\text{Since } \frac{(s\cos s - \sin s)}{s^3} \text{ is an even function}\right]$$

$$\Rightarrow \int_0^{\infty} \frac{(s\cos s - \sin s)}{s^3}\cos\frac{s}{2}\, ds = \frac{-3\pi}{16}$$

(ii) To evaluate $\int_0^{\infty} \frac{(s\cos s - \sin s)}{s^3}\, dx$

Put $x = 0$ in (2) we get $\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}(s\cos s - \sin s)\, ds = 1$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{(s\cos s - \sin s)}{s^3}\, ds = \frac{-\pi}{2}$$

$$\Rightarrow 2\int_0^{\infty} \frac{(s\cos s - \sin s)}{s^3}\, ds = \frac{-\pi}{2} \qquad \text{[Since the integral is even]}$$

$$\Rightarrow \int_0^{\infty} \frac{(s\cos s - \sin s)}{s^3}\, ds = \frac{-\pi}{4}$$

xamples on Infinite Fourier Transform

Show that the Fourier transform of $F(x) = e^{-(x^2)/2}$ is $\sqrt{2\pi}\, e^{-(s^2)/2}$ or show that Fourier transform of $e^{-(x^2)/2}$ is self reciprocal.

By definition of Fourier transform of $F(x)$ we have

$$F\{F(x)\} = \int_{-\infty}^{\infty} F(x)\, e^{isx}\, dx = f(s)$$

$$\Rightarrow \quad F\{F(x)\} = \int_{-\infty}^{\infty} e^{(-x^2)/2}\, e^{isx}\, dx = \int_{-\infty}^{\infty} e^{(-x^2)/2}\, e^{isx} \,.\, e^{(s^2)/2}\, e^{(-s^2)/2}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-is)^2} \,.\, e^{-\frac{s^2}{2}}\, dx = e^{-\frac{s^2}{2}} \,.\, \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-is)^2}\, dx$$

Put $\frac{1}{\sqrt{2}}(x - is) = u, \quad dx\sqrt{2} = du$

Limits : $L\,L$: when $x = -\infty, \quad u = \infty$

$UL: \; x = \infty, \quad u = \infty$

$$\Rightarrow \quad F\{F(x)\} = e^{\frac{-s^2}{2}} \int_{-\infty}^{\infty} e^{-u^2} \sqrt{2}\, du$$

$$= \sqrt{2}\, e^{\frac{-s^2}{2}} \int_{-\infty}^{\infty} e^{-u^2} du,$$

$$= \sqrt{2}\; e^{\frac{-s^2}{2}} \sqrt{\pi} \quad \left[\because \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\right]$$

$$\therefore \quad F\{F(x)\} = \sqrt{2\pi}\; e^{\frac{-s^2}{2}}$$

Hence the Fourier transform of $e^{\frac{-x^2}{2}}$ is $e^{\frac{-s^2}{2}}$

$\Rightarrow \quad F(x)$ is self reciprocal

Find the Fourier transform of $F(x) = \begin{cases} 0 \; -\infty < x < a \\ x, \; a \le x \le b \\ 0, \; x > b \end{cases}$

By definition of Fourier transform, we have $F\{F(x)\} = f(s) = \int_{-\infty}^{\infty} e^{isx}\, F(x)\, dx$

$$\Rightarrow \quad F\{F(x)\} = \int_{-\infty}^{a} e^{isx} \,.\, F(x)\, dx + \int_{a}^{b} e^{isx}\, F(x)\, dx + \int_{b}^{\infty} e^{isx} \,.\, F(x)\, dx$$

$$= \int_{-\infty}^{a} e^{isx} \,.\, 0\, dx + \int_{a}^{b} e^{isx}\, x\, dx + \int_{b}^{\infty} e^{isx} \,.\, 0\, dx$$

$$\left[x\,\frac{e^{isx}}{is} - \frac{e^{isx}}{i^2 s^2}\right]_{x=a}^{b} \Rightarrow \left[\frac{be^{isb}}{is} + \frac{e^{isb}}{s^2}\right] - \left[\frac{ae^{isa}}{is} + \frac{e^{isa}}{s^2}\right]$$

$$\Rightarrow \quad F\{F(x)\} = \frac{i}{s}\left(ae^{isa} - be^{isb}\right) - \left[\frac{e^{isb}}{s^2} - \frac{e^{isa}}{s^2}\right]$$

Given $F(x) = \begin{cases} a\,|-x|, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$

By definition of Fourier transform, we have

$$F\{F(x)\} = f(s) = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} e^{is}\, F(x)\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} e^{isx} F(x)ds + \int_{-a}^{a} e^{isx} F(x)ds + \int_{a}^{\infty} e^{isx} F(x)ds\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 + \int_{-a}^{a} e^{isx}(a - |x|)\,dx + 0\right] = \frac{1}{\sqrt{2\pi}}\left[a\int_{-a}^{a} e^{isx}\,dx - \int_{-a}^{a} |x|\,e^{isx}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[a\int_{-a}^{a} e^{isx}\,dx - \int_{-a}^{0}(-x)\,e^{isx}\,dx - \int_{0}^{a} x e^{isx}\,dx\right] \qquad \left[\text{by definition of } |x| = \begin{cases}-x, & x<0\\ x, & x>0\end{cases}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[a\left(\frac{e^{isx}}{is}\right)_{-a}^{a} + \left(\frac{xe^{isx}}{is} - \frac{e^{isx}}{i^2 s^2}\right)_{x=-a}^{0} - \left(\frac{xe^{isx}}{is} + \frac{e^{isx}}{s^2}\right)_{0}^{a}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a}{is}\,(e^{ias} - e^{-ias}) + \left(\frac{ae^{ias}}{is} + \frac{1}{\sqrt{2}}(1 - e^{ias})\right)_{x=-a}^{0} - \left(\frac{ae^{ias}}{is} + \frac{1}{\sqrt{s^2}}\,(e^{ias} - 1)\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{s^2}\,(2 - 2\cos as)\left[\text{since } \cos as = \frac{e^{ias} + e^{-ias}}{2}\right]\right]$$

$$F\{F(x)\} = f(s) = \sqrt{\frac{2}{\pi}}\left(\frac{1 - \cos as}{s^2}\right)$$

Deduction: By definition of inversion formula of infinite Fourier transform, we have

$$F^{-1}\{f(s)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)\,e^{-isx}\,dx$$

$$\Rightarrow \quad F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left(\frac{1 - \cos as}{s^2}\right)(\cos sx - i\sin sx)\,ds \qquad [e^{isx} = \cos x - i\sin sx]$$

$$\Rightarrow \quad F(x) = \frac{1}{\pi}\left[\int_{-\infty}^{\infty}\left(\frac{1 - \cos as}{s^2}\right)\cos sx\,ds - i\int_{-\infty}^{\infty}\left(\frac{1 - \cos as}{s^2}\right)\sin sx\,ds\right]$$

Equating real and imaginary parts, we get,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1 - \cos as}{s^2}\right)\cos sx\,ds = F(x) \quad \text{and} \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1 - \cos as}{s^2}\right)\sin sx\,ds = 0$$

$$\Rightarrow \quad \frac{2}{\pi}\int_{0}^{\infty}\left(\frac{1 - \cos as}{s^2}\right)\cos sx\,ds = F(x)$$

$$\Rightarrow \quad \int_{0}^{\infty}\left(\frac{1 - \cos as}{s^2}\right)\cos sx\,ds = \frac{\pi}{2}F(x)$$

$$\Rightarrow \int_{0}^{\infty}\left(\frac{1 - \cos as}{s^2}\right)\cos sx\,ds = \begin{cases}\frac{\pi}{2}(a - |x|), & |x| < a\\ 0, & |x| > a\end{cases}$$

Substituting $a = 2$ & $x = 0$ in above, we get,

$$\Rightarrow \int_0^\infty \left[\frac{1-\cos 2s}{s^2}\right] ds = \frac{\pi}{2}(2-0) \qquad \Rightarrow \int_0^\infty \frac{2\sin^2 s}{s^2} ds = \pi \qquad \Rightarrow \int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{2}$$

Find the inverse Fourier Transform of $f(s) = e^{-|s|y}$

Given $f(s) = e^{-|s|y}$

We know that $|s| = \begin{cases} -s, & s < 0 \\ s, & s > 0 \end{cases} \qquad \rightarrow (1)$

By the inversion formula of complex Fourier transform, we have

$$F^{-1}\{f(s)\} = F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\, ds \qquad \Rightarrow \quad F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} e^{-|s|y}\, ds$$

$$\Rightarrow \quad \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-isx} e^{sy}\, ds + \int_{0}^{\infty} e^{-isx} e^{-sy}\, ds\right]$$

$$\Rightarrow \quad \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{(y-ix)s}\, ds + \int_{0}^{\infty} e^{-(y+ix)s}\, ds\right]$$

$$\Rightarrow \quad \frac{1}{2\pi}\left[\left(\frac{e^{(y-ix)s}}{(y-ix)}\right)_{-\infty}^{0} + \left(\frac{e^{-(y+ix)s}}{-(y+ix)}\right)_{0}^{\infty}\right]$$

$$\Rightarrow \quad \frac{1}{2\pi}\left(\frac{1}{(y-ix)} + \frac{1}{(y+ix)}\right) \quad \Rightarrow \frac{1}{2\pi}\left[\frac{2y}{y^2+x^2}\right] = \frac{y}{\pi(y^2+x^2)}$$

Examples on Infinite Fourier Cosine & Sine Transform

Find the Fourier cosine transform of e^{-x^2}

Given $F(x) = e^{-x^2}$

By definition of Fourier cosine transform of $F(x)$, we have

$$F_c\{F(x)\} = f_c(s) = \int_0^\infty f(s)\cos sx\, dx \qquad \text{--- (1)}$$

$$F_c\{e^{-x^2}\} = f_c(s) = \int_0^\infty e^{-x^2}(\cos sx)\, dx \qquad \text{--- (2)}$$

Differentiating (2) w.r.t. 's' we get,

$$\frac{df}{ds} = \frac{d}{ds}\int_0^\infty e^{-x^2}\cos sx\, dx$$

$$\Rightarrow \quad \frac{df}{ds} = \int_0^\infty e^{-x^2}\frac{\partial}{\partial s}\cos sx\, dx$$ [By Lebnitz's Rule of Differentiating under integral sign]

$$\Rightarrow \quad \frac{df}{ds} = \int_0^\infty e^{-x^2}(-x\sin sx)\, dx$$

Multiplying & dividing by 2, we get,

$$= \frac{1}{2}\int_0^\infty (-2xe^{-x^2})\sin sx\, dx$$

$$= \frac{1}{2}\left[e^{-x^2}\sin sx - \int s e^{-x^2}\cos sx\, dx\right]_{x=0}^{\infty}$$

$$= \frac{-s}{2}\int_0^\infty e^{-x^2}\cos sx\, ds \qquad \Rightarrow \frac{df}{ds} = \frac{-s}{2} f$$

This is a first order differential equation, on solving it using variables separable method, we have

$$\Rightarrow \frac{df}{f} = \frac{-s}{2}\, ds$$

On integrating, we get,

$$\int \frac{df}{f} = \int \frac{-s}{2}\, ds \quad \Rightarrow \log f = \frac{-s^2}{4} + \log c \quad \Rightarrow f = c e^{\frac{-s^2}{4}} \qquad \text{--- (3)}$$

To find 'C' put $s = 0$ in (2), we get,

$$f = \int_0^\infty e^{-x^2} dx = \sqrt{\frac{\pi}{2}} \quad \Rightarrow \quad (f)_{s=0} = \sqrt{\frac{\pi}{2}} \qquad \text{--- (4)}$$

Put $s = 0$ in (3) we get, $(f)_{s=0} = c$ --- (5)

From (4) & (5) we have $c = \sqrt{\frac{\pi}{2}}$ $\qquad f_c(s) = \sqrt{\frac{\pi}{2}}\, e^{\frac{-s^2}{4}}$

Hence $F_c\{e^{-x^2}\} = \sqrt{\frac{\pi}{2}}\, e^{\frac{-s^2}{4}}$

Find the Fourier cosine transform of $F(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

Given $F(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

By definition of Fourier Cosine transform, we have

$$F_c\{F(x)\} = \int_0^a \cos x \cos sx\, dx$$

$$F_c\{F(x)\} = f_c(s) = \int_0^\infty F(x)\cos sx\, dx = \int_0^a F(x)\cos sx\, dx + \int_a^\infty F(x)\cos sx\, dx$$

Multiplying & dividing by 2, we get

$$F_c\{F(x)\} = \frac{1}{2}\int_0^a 2\cos sx \cos x\, dx$$

$$\frac{1}{2}\int_0^a [\cos(s+1)x + \cos(s-1)x]dx \qquad [\because 2\cos C\cos D = \cos(C+D) + \cos(C-D)]$$

$$= \frac{1}{2}\left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{s-1}\right]_{x=0}^{a} \qquad = \frac{1}{2}\left[\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s+1}\right]$$

$$\therefore F_c\{F(x)\} = \frac{1}{2}\left[\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s+1}\right]$$

Find the Fourier sine transform of $e^{-|x|}$ and hence evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2} dx$

Given $F(x) = e^{-|x|}$

By definition of Fourier sine transform of $F(x)$, we have

$$F_s\{F(x)\} = f_s(s) = \int_0^\infty F(x) \sin sx \, dx$$

$$\therefore \quad F_s\{e^{-|x|}\} = f_s(s) = \int_0^\infty e^{-|x|} \sin sx \, dx$$

$$\Rightarrow f_s(s) = \int_0^\infty e^{-|x|} \sin sx \, dx$$

$[\because |x| = x \text{ when } x > 0]$

$$\Rightarrow f_s(s) = \left[\frac{e^{-x}}{1+s^2}(-\sin sx - s\cos sx)\right]_{x=0}^{\infty}$$

$$\left[\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b\cos bx]\right]$$

$$\Rightarrow f_s(s) = \left[\frac{e^{-\infty}}{1+s^2}(-\sin(s.\infty) - s\cos(s.\infty)) - \frac{e^0}{1+s^2}(-\sin(s.0) - s.\cos(s.0)\right]$$

$$f_s(s) = 0 - \frac{1}{1+s^2}(0-s)$$

$$\Rightarrow f_s(s) = \frac{s}{1+s^2}$$

To find $\int_0^\infty \frac{x \sin mx}{1+x^2} dx$

By the inversion formula of Fourier sine transform, we have

$$F(x) = F^{-1}\{f_s(s)\} = \frac{2}{\pi}\int_0^\infty f_s(s) \sin sx \, dx$$

$$\Rightarrow \quad F(x) = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+1} \sin sx \, ds = e^{-x}$$

$[\because F(x) = e^{-x} \; x > 0]$

$$\Rightarrow \quad e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{s \sin sm}{1+s^2} ds$$ [on replacing x by m]

$$\Rightarrow \quad e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{x \sin mx}{1+x^2} dx$$ [where 's' replaced by x]

$$\therefore \int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{2}{\pi} e^{-m}$$

nd the Fourier cosine transform of $e^{-ax} \cos ax$.

Given $F(x) = e^{-ax} \cos ax$

By definition of Fourier cosine transform, we have,

$$F_c\{F(x)\} = f_c(s) = \int_0^{\infty} F(x)\cos sx\,dx$$

$$\Rightarrow \quad f_c(s) = \int_0^{\infty} e^{-ax}\cos ax\cos sx\,dx$$

Multiplying and dividing by '2', we get,

$$\Rightarrow \quad f_c(s) = \frac{1}{2}\int_0^{\infty} e^{-ax}(2\cos ax\cos sx)\,dx$$

$$\Rightarrow \quad f_c(s) = \frac{1}{2}\int_0^{\infty} e^{-ax}[\cos(a+s)x + \cos(a-s)x]\,dx \qquad \left[\because 2\cos C\cos D = \cos(C+D) + \cos(C-D)\right]$$

$$f_c(s) = \frac{1}{2}\left[\frac{a}{a^2+(s+a)^2} + \frac{a}{a^2+(s-a)^2}\right] \qquad \Rightarrow \quad f_c(s) = \frac{a}{2}\left[\frac{a^2+(s-a)^2+a^2+(s+a)^2}{[a^2+(s+a)^2][a^2+(s-a)^2]}\right]$$

$$\therefore f_c\{e^{-ax}\cos ax\} = f_c(s) = \frac{a(s^2+2a^2)}{(s^2+2as+2a^2)(s^2-2as+2a^2)}$$

Find the Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$ and hence deduce that

$$\int_0^{\infty} \frac{e^{-ax}-e^{-bx}}{x}\sin sx\,dx = \tan^{-1}\left(\frac{s}{a}\right) - \tan^{-1}\left(\frac{s}{b}\right)$$

Given $F(x) = \frac{e^{-ax}}{x}$

Given $F(x) = \frac{e^{-ax}}{x}$

By definition of Fourier Transform, we have,

$$F_s\{F(x)\} = f_s(s) = \int_0^{\infty} F(x)\sin sx\,dx$$

$$\Rightarrow F_s(s) = \int_0^{\infty} \frac{e^{-ax}}{x}\sin sx\,dx \quad \rightarrow (1)$$

Differentiating (1) both sides with respect to 's', we get,

$$\frac{d}{ds}(f_s(s)) = \frac{d}{ds}\int_0^{\infty} \frac{e^{-ax}}{x}\sin sx\,dx$$

$$\Rightarrow \quad \frac{d}{ds}(f_s(s)) = \int_0^{\infty} \frac{e^{-ax}}{x}\frac{\partial}{\partial s}(\sin sx)\,dx$$

$$\Rightarrow \quad \frac{d}{ds}(f_s(s)) = \int_0^{\infty} \frac{xe^{-ax}\cos sx\,dx}{x} = \int_0^{\infty} e^{-ax}\cos sx\,dx$$

$$\Rightarrow \quad \frac{d}{ds}(f_s(s)) = \frac{a}{s^2+a^2} \qquad \left[\because \int_0^{\infty} e^{-ax}\cos sx = \left(\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\cos sx)\right)_0^{\infty} = \frac{a}{s^2+a^2}\right]$$

Now integrating both sides with respect to s, between 0 to ∞

$$\Rightarrow \quad f_s(s) = \int_0^{\infty} \frac{a}{s^2+a^2}$$

$$\Rightarrow f_s(s) = \tan^{-1}\left(\frac{s}{a}\right) + C \qquad \rightarrow (2)$$

To find C :

Put $s = 0$ in the above equation

$$(F_s(s))_{s=0} = \tan^{-1}\left(\frac{0}{a}\right) + c$$

Put $s = 0$ in the above equation

$$(F_s(s))_{s=0} = \tan^{-1}\left(\frac{0}{a}\right) + c$$

$$(F_s(s))_{s=0} = C \quad \rightarrow (3)$$

Put $s = 0$ in (1) we get,

$$\Rightarrow \quad f_s(s) = \int_0^\infty \frac{e^{-ax}}{x} \sin 0 \, dx = 0$$

$$\Rightarrow \quad f_s(s) = 0 \quad \rightarrow (4)$$

$\therefore$ From (3) & (4) we have $C = 0$

$\therefore$ (2) becomes $F_s\left\{\frac{e^{-ax}}{x}\right\} f_s(s) = \tan^{-1}\left(\frac{s}{a}\right) \quad \rightarrow (5)$

Deduction : Let $F(x) = \frac{e^{-ax} - e^{-bx}}{x}$

By definition of Fourier sine transform of $F(x)$, we have

$$F_s\{F(x)\} = \int_0^\infty F(x) \sin sx \, dx$$

$$F_s\{F(x)\} = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \sin sx \, dx = \int_0^\infty \frac{e^{-ax}}{x} \sin sx - \int_0^\infty e^{-bx} \sin sx \, dx$$

$$F_s\{F(x)\} = \bar{f_s}\left\{\frac{e^{-ax}}{x}\right\} - \bar{f_s}\left\{\frac{e^{-bx}}{x}\right\}$$

$$= \tan^{-1}\left[\frac{s}{a}\right] - \tan^{-1}\left[\frac{s}{b}\right] \quad \text{[by (5)]}$$

$$\Rightarrow \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \sin sx \, dx = \tan^{-1}\left(\frac{s}{a}\right) - \tan^{-1}\left(\frac{s}{b}\right)$$

Find the Fourier cosine transform of $F(x) = \frac{e^{-ax}}{x}$

Given $F(x) = \frac{e^{-ax}}{x}$

By definition of Fourier cosine transform of $F(x)$, we have,

$$F_c\{F(x)\} = f_c(s) = \int_0^\infty F(x) \cos sx \, dx \quad \Rightarrow \quad F_c(s) = \int_0^\infty \frac{e^{-ax}}{x} \cos sx \, dx \quad \rightarrow (1)$$

Differentiating (1) w.r.t. 's',

$$\frac{d}{ds}(f_c(s)) = \frac{d}{ds}\int_0^\infty \frac{e^{-ax}}{x} \cos sx \, dx$$

[By Leibnitz's Rule of differentiating under the integral sign]

$$\Rightarrow \frac{d}{ds}(f_c(s)) = \int_0^\infty \frac{e^{-ax}}{x} (-\sin sx) \times dx$$

$$\Rightarrow \frac{d}{ds}(f_c(s)) = -\int_0^\infty e^{-ax} \sin sx \, dx = \frac{-s}{s^2 + a^2} \qquad \left[\because \int_0^\infty e^{-ax} \sin dx = \frac{b}{a^2 + b^2}\right]$$

On integrating both sides, we get, $\Rightarrow \quad F_c\{F(x)\} = f_c(s) = -\int \frac{s}{s^2 + a^2} ds$

$$\Rightarrow F_c\{F(x)\} = -\frac{1}{2}\log(s^2+a^2)$$

7. Find Fourier cosine and sine transform of e^{-ax}, $a > 0$ and hence deduce the inversion formula (or) hence deduce the integrals (i) $\int_0^\infty \frac{\cos sx}{s^2+a^2}\,ds$ (ii) $\int_0^\infty \frac{s\sin sx}{s^2+a^2}\,ds$

Given $F(x) = e^{-ax}$, $a > 0$

To find Fourier cosine transform:

By the definition of Fourier cosine transform of $F(x)$, we have,

$$F_c\{F(x)\} = f_c(s) = \int_0^\infty F(x)\cos sx\,dx$$

$$= \int_0^\infty e^{-ax}\cos sx\,dx$$

$$\left[\frac{e^{-ax}}{a^2+b^2}(-a\cos sx + s\sin sx)\right]_{x=0}^\infty$$

$$\left[\because \int_0^\infty e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \left[\frac{e^{-\infty}}{a^2+s^2}(-a\sin\infty - s\cos\infty)\frac{-e^0}{a^2+s^2}(-a\sin 0 - s\cos 0)\right]$$

$$\Rightarrow \boxed{F_s(s) = \frac{s}{s^2+a^2}}$$

Deduction: (1) By inversion formula of Fourier cosine transform we have,

$$F^{-1}\{f_c(s)\} = F(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{s^2+a^2}\cos sx\,ds \qquad [\because F(x) = e^{-ax}]$$

$$\Rightarrow \frac{\pi e^{-ax}}{2a} = \int_0^\infty \frac{\cos sx}{s^2+a^2}\,ds$$

i.e., $\int_0^\infty \frac{\cos sx}{s^2+a^2}\,ds = \frac{\pi e^{-ax}}{2a}$

By the inversion formula of Fourier sine transform we have,

$$F^{-1}\{f_s(s)\} = F(x) = \frac{2}{\pi}\int_0^\infty f(s)\sin sx\,ds \qquad \Rightarrow F(x) = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+a^2}\sin sx\,ds$$

$$\Rightarrow e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+a^2}\sin sx\,ds \qquad \Rightarrow \boxed{\int_0^\infty \frac{s}{s^2+a^2}\sin sx\,ds = \frac{\pi}{2}e^{-ax}}$$

Find Fourier cosine transform of $F(x) = \frac{1}{1+x^2}$ and hence find the Fourier sine transform of $F(x) = \frac{1}{1+x^2}$

Given $F(x) = \frac{1}{1+x^2}$ By the definition of Fourier cosine transform of $F(x)$ we have,

$$F_c\{F(x)\} = f_c(s) = \int_0^\infty F(x)\cos sx\,ds = \int_0^\infty \frac{1}{1+x^2}\cos sx\,dx \quad \to (1)$$

Differentiating (1) w.r.t. 's', we have

$$\frac{d(f_c(s))}{ds}=\frac{d}{ds}\int_0^\infty \frac{1}{1+x^2}\cos sx\,dx \qquad \Rightarrow \frac{df}{ds}=-\int_0^\infty \frac{x\sin sx}{1+x^2}\,dx$$

Multiplying and dividing by 'x', we get,

$$\Rightarrow \frac{df}{ds}=-\int_0^\infty \frac{x^2\sin sx}{x(1+x^2)}\,dx \quad \Rightarrow \frac{df}{ds}=-\int_0^\infty \frac{(x^2+1-1)}{x(1+x^2)}\sin sx\,dx$$

$$\Rightarrow \frac{df}{ds}=-\int_0^\infty \frac{(x^2+1)}{x(1+x^2)}\sin sx\,dx+\int_0^\infty \frac{1}{x(1+x^2)}\sin sx\,dx=\frac{\pi}{2}+\int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx \quad \rightarrow(2)$$

Differentiating (2) with respect to 's' we get, $\frac{d^2f}{ds^2}=0+\int_0^\infty \frac{\cos sx}{1+x^2}\,dx$

$$\Rightarrow \frac{d^2f}{ds^2}=f_c\{f(x)\} \qquad \text{[by (1)]}$$

$$\Rightarrow \frac{d^2f}{ds^2}=f_c(s) \qquad \Rightarrow \frac{d^2f}{ds^2}-f=0 \quad \Rightarrow (D^2-1)f=0 \quad \rightarrow(3)$$

Where $D=\frac{d}{ds}$

This is a 2[nd] order Linear differential equation with constant coefficients, its general solution is given by

$$f=c_1e^s+c_2e^{-s} \qquad \rightarrow(4)$$

To find C_1 & C_2:

put $s=0$ in (4), we get

$$[f_c(s)]_{s=0}=C_1+C_2 \qquad \text{--- (5)}$$

Put $s=0$ in (1) we get $[f_c(s)]_{s=0}=\int_0^\infty \frac{1}{1+x^2}\cos 0\,dx=[\tan^{-1}x]_0^\infty=\frac{\pi}{2}$

From (5) & (6), we have, $C_1+C_2=\frac{\pi}{2}$

Now differentiating (4) w.r.t 's', we have

$\frac{df}{ds}=c_1e^s-c_2e^{-s} \rightarrow(8)$ now put $s=0$ in (8), we get

$\left(\frac{df}{ds}\right)_{s=0}=C_1-C_2 \quad \rightarrow (9)$ also put $s=0$ in (2) we get

$$\left(\frac{df}{ds}\right)_{s=0}=\frac{\pi}{2}+0 \quad \Rightarrow \left(\frac{df}{ds}\right)_{s=0}=\frac{\pi}{2} \quad \rightarrow(10)$$

From (9) & (10), we get, $C_1-C_2=\frac{\pi}{2} \quad \rightarrow (11)$

Solving (7) & (11), we get, $C_1=0$ & $C_2=\frac{\pi}{2}$

substituting C_1 & C_2 in (4) $F_c\{F(x)\}=f_c(s)=\frac{\pi}{2}e^{-s}$

Deduction: Consider $f_c(s)=\frac{\pi}{2}e^{-s}$

Differentiating w.r.t 's', we get $\Rightarrow \frac{d[f_c(s)]}{ds}=\frac{\pi}{2}e^{-s}$

Differentiating w.r.t 's', we get $\Rightarrow \frac{d[f_c(s)]}{ds} = \frac{\pi}{2} e^{-s}$

$$F_s\left\{\frac{-x}{1+x^2}\right\} = \frac{-\pi}{2} e^{-s}$$

$$\Rightarrow F_s\left\{\frac{x}{1+x^2}\right\} = \frac{-\pi}{2} e^{-s}$$

Exercise Problems :

1. Find the Fourier transform of $F(x) = \begin{cases} e^{-as}, & a < x < b \\ 0, & x < a \,,\, x < b \end{cases}$
2. Find the Fourier transform of $F(x) = \{\sin x, 0 < x < \pi$ and 0, otherwise
3. Find the Fourier sine transform of $F(x) = \frac{1}{x(x^2+a^2)}$ and hence deduce the cosine transform of $\frac{1}{x(x^2+a^2)}$
4. Find the Fourier Cosine & Sine transforms of $F(x) = 2e^{-5x} + 5e^{-2x}$
5. Prove that the Fourier sine transform of $F(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$ is $\frac{2\sin s\,(1-\cos s)}{s^2}$
6. Find the inverse Fourier sine transform of $f_s(s) = \frac{s}{1+s^2}$

FINITE FOURIER TRANSFORMS:-

If f(x) is a function defined in the interval (0,c) then, the Finite Fourier sine transform of f(x) in 0<x<c is defined as

$$F_s(n) = \int_0^c f(x) \sin\frac{n\pi x}{c}\, dx$$, where n is an integer.

The Inverse finite Fourier sine transform of $F_s(n)$ is f(x) and is given by

$$f(x) = \frac{2}{c} \sum_{n=1}^{\infty} F_s(n) \sin\frac{n\pi x}{c}$$

The Finite Fourier cosine transform of f(x) in 0<x<c is defined as

$$F_c(n) = \int_0^c f(x) \cos\frac{n\pi x}{c}\, dx$$, where n is an integer

The Inverse finite Fourier cosine transform of $F_c(n)$ is f(x) and is given by

$$f(x) = \frac{1}{c} F_c(0) + \frac{2}{c} \sum_{n=1}^{\infty} F_c(n) \cos\frac{n\pi x}{c}$$

Problems:-

1. Find the Finite Fourier sine and cosine transforms of f(x)=1 in (0,c)

sol: By definition,

$$F_s(n) = \int_0^c f(x) \sin\frac{n\pi x}{c}\, dx$$

$$= \int_0^c \sin\frac{n\pi x}{c}\, dx$$

$$= [\left(\frac{-c}{n\pi}\right)\cos\left(\frac{n\pi x}{c}\right)]_0^c$$

$$= \frac{-c}{n\pi}\,(\cos n\pi - 1)$$

$$= \frac{c}{n\pi}(1 - (-1)^n)$$

$$F_s(n) = \begin{cases} 0, \text{if } n \text{ is odd} \\ \frac{2c}{n\pi}, \text{if } n \text{ is even} \end{cases}$$

Now, $F_c(n) = \int_0^c f(x) \cos\frac{n\pi x}{c}\, dx$

$$= \int_0^c \cos\frac{n\pi x}{c}\, dx$$

$$= [\left(\frac{c}{n\pi}\right)\sin\left(\frac{n\pi x}{c}\right)]_0^c$$

$$= \frac{c}{n\pi}\, \sin n\prod\ = 0$$

xamples on Finite Fourier Sine and Cosine Transforms

Find the finite cosine transform of $F(x) = \frac{\pi}{2} - x + \frac{x^2}{2\pi}$ where $0 < x < \pi$

Given $F(x) = \frac{\pi}{2} - x + \frac{x^2}{2\pi}$

By definition of finite cosine transform $F_c\{F(s)\} = f_c(s) = \int_0^\pi F(x) \cos\frac{s\pi x}{\pi}$ [$here\ l = \pi$]

$$\Rightarrow F_c\{F(x)\} = \int_0^\pi f(x) \cos sx\, ds$$

$$\Rightarrow F_c\left[\frac{\pi}{2} - x + \frac{x^2}{2\pi}\right] = f_c(s) = \int_0^\pi \left[\frac{\pi}{2} - x + \frac{x^2}{2\pi}\right] \cos sx\, dx$$ on intergrating by parts we have

$$F_c\left[\frac{\pi}{2} - x + \frac{x^2}{2\pi}\right]\frac{\sin sx}{s} - \frac{1}{s}\int_0^\pi \left[-1 + \frac{x}{\pi}\right] \sin sx\, dx$$

$$\Rightarrow f_c(s) = \left[\left[\frac{\pi}{2} - x + \frac{x^2}{2\pi}\right] = \frac{\sin sx}{s} - \left[\frac{\pi}{2} - 0 + \frac{0^2}{2\pi}\right]\frac{\sin 0}{s}\right] - \frac{1}{s}\int_0^\pi \left[\frac{x}{\pi} - 1\right] \sin x$$

$$\Rightarrow f_c(s) = (0 - 0) - \frac{1}{s}\left[\left[\frac{x}{\pi} - 1\right]\left[\frac{-\cos sx}{s}\right] \int \frac{1}{\pi}\frac{(-\cos sx)}{s}\, dx\right]$$

$$f_c(s) = \frac{-1}{s}\left[-\left[\frac{x}{\pi} - 1\right]\frac{\cos sx}{s} + \frac{1}{s^2\pi}\sin sx\right]_{x=0}^{\pi}$$

$$\Rightarrow f_c(s) = \frac{-1}{s}\left[-\left[\frac{\pi}{\pi} - 1\right]\frac{\cos s\pi}{s} + \frac{1}{s^2\pi}\sin s\pi\right]_{x=0}^{\pi} + \left[\frac{0}{\pi} - 1\right]\frac{\cos 0}{s}\frac{-1}{s^2\pi}\sin 0\right]$$

$\Rightarrow f_c(s) = \frac{-1}{s}\left[0 + 0 - \frac{1}{s} - 0\right]$ $\quad \Rightarrow f_c(s) = \frac{1^2}{s}$ $\quad$ if $s = 1, 2, 3 \ldots$

2. Find the inverse finite sine transform of $F(x)$ if $f_c(s) = \frac{1-\cos s\pi}{s^2\pi^2}$ where $0 < x < \pi$

Given $f_c(s) = \frac{1-\cos s\pi}{s^2\pi^2}$ in $0 < x < \pi$

By the inverse foureir sine transform, we have,

$$F(x) = \frac{2}{l}\sum_{n-1}^{\infty} f_s(s)\sin\left[\frac{s\pi x}{l}\right] \Rightarrow = \frac{2}{l}\sum_{n-1}^{\infty}\left[\frac{1-\cos s\pi}{s^2\pi^2}\right]\sin sx$$

$$F(x) = \frac{2}{l}\sum_{n-1}^{\infty}\left[\frac{1-\cos s\pi}{s^2\pi^2}\right]\sin sx$$

Find the finite cosine transform of $F(x)$ defined by $F(x) = \left[1-\frac{x}{\pi}\right]^2$ where $0<x<\pi$

Given $F(x) = \left[1-\frac{x}{\pi}\right]^2$ where $0<x<\pi$

By finite Fourier sine transform of $F(x)$, we have, $F_s\{F(x)\} = f_s(S) = \int_0^l F(x)\sin\frac{s\pi x}{l}dx$

$\Rightarrow f_c(s) = \int_0^l\left[1-\frac{x}{\pi}\right]^2 \sin sx\, dx$ $[\therefore l = \pi]$ on integrating by parts, we get

$$\Rightarrow f_c(s) = \left[\left[1-\frac{x}{\pi}\right]\frac{\cos sx}{s}\right]_0^{\pi} - \int_0^{\pi}\frac{\cos sx}{s}2\left[1-\frac{x}{\pi}\right]\left[-\frac{1}{\pi}\right]dx$$

$$\Rightarrow f_c(s) = \left[-\left[1-\frac{\pi}{\pi}\right]\frac{\cos s\pi}{s} + \left[1-\frac{0}{\pi}\right]\frac{\cos}{s}\right] + \frac{2}{\pi s}\int_0^{\pi}\left[1-\frac{x}{\pi}\right]\cos sx\, dx$$

Again integrating by parts we have

$$\Rightarrow f_c(s)=\left[0+\frac{1}{\pi}\right]+\frac{2}{\pi s}\left[\left[1-\frac{x}{\pi}\right]\frac{\sin sx}{s}-\int\frac{-1}{\pi}\frac{\sin sx}{s}\,dx\right]$$

$$\Rightarrow f_c(s)=\frac{1}{s}+\frac{2}{\pi s}\left[\left[1-\frac{x}{\pi}\right]\frac{\sin sx}{s}-\frac{-1}{\pi s^2}\cos sx\right]_{x=0}^{\pi}$$

$$\Rightarrow f_c(s)=\frac{1}{s}+\frac{2}{\pi s}\left[\left[1-\frac{\pi}{\pi}\right]\frac{\sin s\pi}{s}-\frac{-1}{\pi s^2}\cos s\pi\left[1-\frac{0}{\pi}\right]\frac{\sin 0}{s}+\frac{1}{s^2\pi}\right]$$

$$\Rightarrow f_c(s)=\frac{1}{s}+\frac{2}{\pi s}\left[0-\frac{(-1)^s}{s^2\pi}-0+\frac{1}{s^2\pi}\right] \qquad \Rightarrow f_c(s)=\frac{1}{s}+\frac{2}{s^3\pi}\left[1-(-1)^s\right]$$

Exercise Problems:

1. Find the finite Fourier cosine transform of $F(x)=\begin{cases} x, & 0<x<\frac{\pi}{2} \\ \pi-x, & \frac{\pi}{2}<x<\pi \end{cases}$

2. Find the finite Fourier sine & cosine transform of $F(x)=x(\pi-x)$ in $0<x<\pi$

3. Find the finite Fourier cosine transform of $F(x)=\frac{x^2}{2\pi}-\frac{\pi}{2}$, $\quad 0\le\pi$

Short Type Question and Answers

Problem 1 Write the formula for finding Euler's constants of a Fourier series in (0, 2 π)

Solution:

Euler's constants of a Fourier series in (0, 2 π) is given by

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Problem 2 Write the formula for Fourier constants for f(x) in the interval $(-\pi,\pi)$.

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Problem 3: Find the constant a_0 of the Fourier series for the function f(x) = k , [0 , 2∏]

Solution : $a_0 = \frac{1}{\Pi}\int_0^{2\Pi} f(x)dx = = \frac{1}{\Pi}\int_0^{2\Pi} k\,dx = \frac{1}{\Pi}(\,kx\,)_0^{\,2\Pi}$

= 2k.

Problem 4 If $f(x) = e^x$ in $-\pi < x < \pi$, find a_n.

Solution:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\frac{e^x}{1+n^2}(\cos nx + n\sin nx)\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\frac{e^{\pi}}{1+n^2}(-1)^n - \frac{e^{-\pi}}{1+n^2}(-1)^n\right\}$$

$$a_n = \frac{(-1)^n}{\pi(1+n^2)}\left\{e^{\pi} - e^{-\pi}\right\}.$$

Problem 5 Write the formula's for Fourier constants for f(x) in (c, c+2*l*).

Solution:

$$a_0 = \frac{1}{\ell}\int_{c}^{c+2\ell} f(x)\,dx$$

$$a_n = \frac{1}{\ell}\int_{c}^{c+2\ell} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\ell}\int_{c}^{c+2\ell} f(x)\sin nx\,dx$$

Problem 6 Write the formulas for Fourier constants for f(x) in (-*l*, *l*).

Solution:

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin nx\,dx$$

Problem 7 : What is the sum of Fourier series at a point $x = x_0$, where the function f(x) has a finite discontinuity ?

Solution : Sum of Fourier series at a point $x = x_0$ is

$$\frac{f(x_0^+) + f(x_0^-)}{2}$$

Problem 8 If $x^2 = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} ....to\infty\right) \quad -(1)$

in $-\pi \le x \le \pi$, find $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...to \ \infty$.

Solution:

Put $x = \frac{\pi}{2}$ a point of continuity

$$\therefore (1) \quad \Rightarrow \frac{\pi^2}{4} = \frac{\pi^2}{3} - 4\left\{\frac{1}{1^2} + \frac{1}{2^2} + ...\right\}$$

$$\Rightarrow \frac{\pi^2}{4} - \frac{\pi^2}{3} = -4\left\{\frac{1}{1^2} + \frac{1}{2^2} + ...\infty\right\}$$

$$\frac{-\pi^2}{12} \cdot \frac{1}{-4} = \frac{1}{1^2} + \frac{1}{2^2} + ...\infty$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...\infty = \frac{\pi^2}{48}$$

Problem 9 Check whether the function is odd or even, where f(x) is defined by

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi} & -\pi < x < 0 \\ 1 - \frac{2x}{\pi} & 0 < x < \pi \end{cases}$$

Solution:

$$For -\pi < x < 0,\ f(-x) = 1 + \frac{2(-x)}{\pi} = 1 - \frac{2x}{\pi} = f(x), where\ 0 < x < \pi$$

$\Rightarrow f(x)$ is an even function.

Problem 10 When an even function f(x) is expanded in a Fourier series in the interval $-\pi < x < \pi$, show that $b_n = 0$.

Solution:

$$b_n = \frac{1}{n}\int_{-n}^{n} f(x)\sin nx dx$$

Given $f(x)$ is even and

$\sin nx$ is odd function

Even x Odd = Odd .Therefore f(x)sin nx is odd function.

$b_n = 0$.

Problem 11 Find the Fourier constant b_n for x sin x in $-\pi < x < \pi$, when expressed as a Fourier series.

Solution:

$$f(x) = x\sin x$$

$$f(-x) = -x\sin(-x)$$

$$= x\sin x = f(x)$$

Here $f(x)$ is an even function

$\therefore b_n = 0$

Problem 12 If f(x) is a function defined in $-2 \le x \le 2$, what is the value of b_n ?

Solution:

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx$$

Problem 13 Explain half range cosine series in $(0,\pi)$.

Solution:

Half range cosine series in $(0,\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos n\pi$$

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nxdx$$

Problem 14 Find the sine series of f(x) = k in $(0,\pi)$.

Solution:

$$f(x) = \sum_{1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} k\sin nxdx$$

$$b_n = \frac{2k}{\pi}\left\{\frac{1-(-1)^n}{n}\right\}$$

Therefore f(x) = $\sum_{n=1}^{\infty} \frac{2k}{\pi}\left[\frac{1-(-1)^n}{n}\right]$ sin nx .

Problem 15 : Write Parseval's formula in the interval (c , c + 2n)

Solution :

$$\frac{1}{2\pi}\int_{c}^{c+2\pi} (f(x))^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{1}^{\infty}(a_n^2 + b_n^2)$$

FOURIER TRANSFORMS:

Problem 1 If the Fourier transform of $f(x)$ is $F(s)$ then, what is Fourier transform of $f(ax)$?

Solution:

Fourier transform of $f(x)$ is

$$F(s) = F(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F(f(ax)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{isx}dx$$

Put $t = ax$

$dt = adx$

$$F(f(ax)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{is(t/a)}\frac{dt}{a}$$

$$= \frac{1}{a}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist/a}dt$$

$$= F\left(f(ax)\right) = \frac{1}{a}.F\left(\frac{s}{a}\right).$$

Problem 2 Find the Fourier sine transform of e^{-3x}

Solution:

$$F_s\left(f(x)\right) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$$

$$F_s\left(e^{-3x}\right) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-3x} \sin sx \, dx$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{s \sin sx}{s^2 + a^2} ds$$

$$\frac{\pi}{2} f(x) = \int_0^{\infty} \frac{s \sin sx}{s^2 + a^2} ds$$

$$\frac{\pi}{2} e^{-ax} = \int_0^{\infty} \frac{s \sin sx}{s^2 + a^2} ds$$

Put $a = 1, \quad x = \alpha$

$$\frac{\pi}{2} e^{-\alpha} = \int_0^{\infty} \frac{s \sin sx}{s^2 + 1} ds$$

Replace 's' by 'x'

$$\int_0^{\infty} \frac{s \sin sx}{1 + x^2} dx = \frac{\pi}{2} e^{-\alpha}.$$

Problem 3 Find the Fourier sine transform of $f(x) = e^{-ax}$, $a > 0$. Hence deduce that $\int_0^{\infty} \frac{x \sin \alpha x}{1 + x^2} dx = \frac{\pi}{2} e^{-\alpha}$.

Solution:

$$F_s\left(f(x)\right) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$$

$$F_s\left(e^{-ax}\right) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2 + a^2}\right)$$

By inverse Sine transform, we get

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(s) \sin sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2 + a^2}\right) \sin sx \, ds$$

Problem 4 Prove that $F_C\left[f(x)\cos ax\right]=\frac{1}{2}\left[F_C(s+a)+F_C(s-a)\right]$

Solution:

$$F_c(s)=F_c\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c\left[f(x)\cos ax\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ax\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\left[\frac{\cos(a+s)x+\cos(a-s)x}{2}\right]dx$$

$$=\frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s+a)xdx\right\}+\frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)xdx\right\}$$

$$=\frac{1}{2}\left[F_c(s+a)+F_c(s-a)\right]$$

Problem 5 : Find the Fourier cosine transform of f(x) = $\begin{cases}\cos x, & 0<x<a\\ 0, & x\geq a\end{cases}$

Solution :

$$F_c(f(x))=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx=\sqrt{\frac{2}{\pi}}\int_0^a \cos x\cos sx dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^a\left[\frac{\cos(s+1)x+\cos(s-1)x}{2}\right]dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)x}{s+1}+\frac{\sin(s-1)x}{s-1}\right]_0^a$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)a}{s+1}+\frac{\sin(s-1)a}{s-1}\right],\text{ provided } S\neq 1,\ S\neq -1.$$

Problem 6 Find $F_C\left(xe^{-ax}\right)$ and $F_S\left(xe^{-}\right.$

Solution:

$$F_c\left(xe^{-ax}\right)=\frac{d}{ds}F_s\left[f(x)\right]$$

$$F_c\left(xe^{-ax}\right)=\frac{d}{ds}F_s\left[e^{-ax}\right]$$

$$=\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx\right]$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{\left(s^2+a^2\right)^2}\right].$$

$$F_s\left[xe^{-ax}\right] = -\frac{d}{ds}\left[F_c e^{-ax}\right]\left(\therefore F_s\left(xf(x)\right) = -\frac{d}{ds}\left(F_c\left(f(x)\right)\right)\right)$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx\right]$$

$$= -\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{2as}{\left(s^2+a^2\right)^2}\right].$$

Problem 7 If $F(s)$ is the Fourier transform of $f(x)$, then prove that the Fourier transform of $e^{ax}f(x)$ is $F(s+a)$.

Solution:

$$F(s) = F\left(f(x)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F\left(e^{iax}f(x)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax}f(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(a+s)x}f(x)\,dx$$

$$= F(s+a).$$

Problem 8 : Define Convolution Theorem.

Solution :

If F(s) and G(s) are Fourier transform of $f(x)$ and $g(x)$ respectively, Then the Fourier transform of the convolutions of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

i.e. $F\left[f(x)*g(x)\right] = F\left[f(x)\right]F\left[g(x)\right]$

Problem 9 : Derive the relation between Fourier transform and Laplace transform**.**

Solution:

Consider $f(t)\begin{cases} e^{-xt}g(t), & t>0 \\ 0 \quad , & t<0 \end{cases} \qquad -(1)$

The Fourier transform of $f(x)$ is given by

$$F\left[f(t)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-xt}g(t)e^{ist}dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(is-x)t} g(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-pt} g(t)\, dt \text{ where } p = x - is$$

$$= \frac{1}{\sqrt{2\pi}} L(g(t)) \left[\therefore L\left(f(t) = \int_{0}^{\infty} e^{-st} f(t)\, dt \right) \right]$$

$\therefore$ Fourier transform of $f(t) = \frac{1}{\sqrt{2\pi}} \times$ Laplace transform of g(t) where g(t) is defined by (1).

Problem 10 : Find the Fourier sine Transform of $\frac{1}{x}$.

Solution:

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx$$

$$F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{x} \sin sx\, dx$$

Let $sx = \theta$

$sdx = d\theta;\ \theta : 0 \to \infty$

$$F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{s}{\theta} \sin\theta \frac{d\theta}{s}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin\theta}{\theta}\, d\theta \left[\therefore \int_0^{\infty} \frac{\sin\theta}{\theta} d\theta = \frac{\pi}{2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\pi}{2}\right) = \sqrt{\frac{\pi}{2}}.$$

Problem 11 : Find the Fourier sine Transform of f(x) = $\begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

Solution:

The Fourier sine transform of $f(x)$ is given by $F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx$

$$= \sqrt{\frac{2}{\pi}} \left\{ \int_0^1 \sin sx\, dx + \int_1^{\infty} 0 \sin sx\, dx \right\} = \sqrt{\frac{2}{\pi}} \left[\frac{-\cos sx}{s} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{-\cos s}{s} + \frac{1}{s} \right\} = \sqrt{\frac{2}{\pi}} \left[\frac{1}{s} - \frac{\cos s}{s} \right].$$

UNIT V : Partial Differential Equations

Non- Linear Equations of First order

A partial differential equation of first order but of degree more than one is called a non-linear partial differential equation.

Standard Form I:

Equations involving only p,q and not x,y,z.

i.e f(p,q) = 0 --------(1)

an integral of (1) is given by

$z=ax+by+c$-----------(2)

where a and b are connected by the relation

$f(a,b)=0$----(3)

since from (2) $p=\frac{\partial z}{\partial x}=a$ and $\frac{\partial z}{\partial y}=b$

which when substituted in (3) yields (1)

i.e (2) satisfies the given equation

now solving (3) for b, let b =F(a). putting this value of b in (2), the complete integral is given by

$z= ax+y F(a)+c$ -----------(4)

The singular integral is obtained by eliminating a and c between the complete integral (4) and the equations obtained by differentiating (4) w.r.t 'a' and c.

Standard Form IV:

$Z=px+qy+f(a,b)$

Clairaut's Type:

Equations of this type have form

$Z=px+qy+f(p,q)$--------(1)

We can easily verify that a solution 1 is

$Z=ax+by+f(a,b)$-------------(2)

Where a, b are arbitrary constants, therefore it is the complete integral.

Partially differentiating (2) w.r.t a and b in turn and equating to zero the results derived, we have the equations.

0= x+of/oa--------(3)

And 0= y+of/ob----------------(4)

Eliminating a and b from the equations (2), (3) and (4) we get singular solution.

To obtain the general integral, we put $b = \phi(a)$ in (2), where ϕ is an arbitrary function.

Then $z= ax+y \phi(a)+f[a, \phi(a)]$ --------------(5)

Partially differentiating (5) w.r.t a and equating it to zero we get

$0= x+y \phi^1(a)+f^1(a)$ --------------------(6)

The elimination of a between the equations (5) and (6) is the general integral.

Standard Form II:

Equation does not involve x and y

i.e $f(z,p,q) = 0$ ------------------(1)

we take $q= ap$ -------------(2)

where a is an orbitary constant.

Solve (1) and (2) for p in terms of z say, we obtain

$P=\phi(z)$----------------(3)

$dz = pdx+qdy$

= pdx+a pdy

=p(ax+ady)

dx+ady = dz/ϕ(z) -------------------(4)

integrating (4),

$$x+ay = \int \frac{dz}{\phi(z)} + b \text{ ----------(5)}$$

which is the complete integral of (1) working rule of solve f(p,q,z)= 0;

1. Let us assume u = x+ay and using p= dz/du and q = adz/du in the given equation f(z,p,q) = 0 and which transform into f(z,dz/du, adz/du) = 0.
2. Solve the resulting ordinary differential equation f(z,dz/du, adz/du) =0
3. Substituting x+ay in place of u.

STANDARD FORM III. VARIABLES SEPARABLE

Equation of the form f_1 (x,p) = f_2 (y,q) i.e. equations not involving z and the terms containing x and p can be separated from those containing y and q.

As a trail solution, we assume each side equal to an arbitrary constant a, solve for p and q from the resulting equation.

$f_1(x,p)$ = a and $f_2(x,p)$=a

Solving for p and q, we obtain

P = $F_1(x,a)$ and q= F_2 (y,a)

Since z is a function of x and y, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx = qdy$$

$$\therefore dz = F_1(x,a)dx + F_2(y,a)dy + b$$

Integrating z = $\int F_1(x,a)dx + \int F_2(y,a)dy + b$

Which is the required complete solution containing two arbitrary constants a and b.

Example : Solve p –q = x^2+y^2

Solution: Seperating p and x from q and y, the given equation can be written as p-x^2=q+y^2=a, (say)

$\therefore\ p - x^2 = a$ gives p = a+x^2 and q=y^2=a gives q=a-y^2

Putting the values of p and q and dz = pdx + qdy, we get

dz= (a=x^2) dx+ (a-y^2) dy

Integrating z = ax+ $\frac{x^3}{3} + ay - \frac{y^3}{3} + b = \frac{1}{3}(x^2 - y^3) + a(x+y) + b$

Which is the desired solution.

Example : Solve p^2+q^2 = x^2+y^2

Solution: Given equation can be written as

p^2- x^2 = y^2- q^2 = a, say

$$\therefore p^2 - x^2 = a \Rightarrow p = \sqrt{x^2 + a}$$

and $y^2 - q^2 = a \Rightarrow p = \sqrt{y^2 - a}$

Substituting these values of p and q in dz= pdx + qdy, we get

$$dz = \sqrt{x^2 + a}dx + \sqrt{y^2 - a}dy$$

Integrating,we get

$$\int dz = \int \sqrt{x^2 + \left(\sqrt{a}\right)^2}dx + \int \sqrt{y^2 - \left(\sqrt{a}\right)^2}dy$$

$$\Rightarrow z = \frac{x}{2}\sqrt{x^2 + a} + \frac{a}{2}\sinh^{-1}\frac{x}{\sqrt{a}} + \frac{y}{2}\sqrt{y^2 - a} - \frac{a}{2}\cosh^{-1}\frac{x}{\sqrt{a}} + c$$

$$= \frac{1}{2}\left(x\sqrt{x^2 + a} + y\sqrt{y^2 - a}\right) + \frac{a}{2}\left(\sinh^{-1}\frac{x}{\sqrt{a}} - \cosh^{-1}\frac{x}{\sqrt{a}}\right) + c$$

Which is the required solution

ONE DIMENSIONAL WAVE EQUATION

Let OA be a stretched string of length l with fixed ends O and A. Let us take x-axis along OA and y-axis along OB perpendicular to OA, with O as origin. Let us assume that the tension T in the string is constant and large when compares with the string so that the effects of gravity are negligeable. Let us pluck the string in the BOA plane and allow it to vibrate. Let p be any point of the string at time t. Let there be no external forces acting on the string. Let each point of the string make small vibrations at right angles to OA in the plane of BOA. Draw pp^1 perpendicular to OA. Let $op^1 = x$ and $pp^1 = y$. Then y is a function of x and t. Under the assumptions, using Newton's Second Law of motion, it can be proved that $y(x,t)$ is governed by the equation,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - - - - - - - - - - - (1)$$

$$i,e., \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$$where \ c^2 = T/m$$

With T = tension in the string at any point and m is mass per unit length of the string.

Since the points O and A are not disturbed from their original positions for any time t we get

$$y(0,t) = 0 - - - - - - (2)$$

$$y(1,t) = 0 - - - - - - (3)$$

These are referred to as the end conditions or boundary conditions. Further it is possible that, we describe the initial position of the string as well as the initial velocity at any point of the string at time $t = 0$ through the conditions

$$y(x,O) = f(x), 0 \le x \le l ------(4)$$

$$\frac{\partial y}{\partial t}(x,O) = g(x), 0 \le x \le l -----(5)$$

Where $f(x)$ and $g(x)$ are functions such that $f(O) = f(l) = 0$;and $g(O) = g(l) = 0$. Thus to study the subsequent motion of any point of the string we have to solve following :

Determine $y(x,t)$ such that $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} -----(1)$

Subject to the condition

$$\left.\begin{array}{l} y(O,t) = 0 \;\; for\ all\ t ----(2) \\ y(l,t) = 0 \;\; for\ all\ t ----(3) \end{array}\right\} end\ condtions$$

$$\left.\begin{array}{l} y(x,O) \;\; = f(x), O \le x \le 1 ----(4) \\ \left(\frac{\partial y}{\partial t}\right)_{at\ t=0} = g(x), O \le x \le 1 ----(5) \end{array}\right\} initial\ condtions$$

The equation (1) is called one dimensional wave equation

Solution of equation (1) to (5)

Consider the equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} -----(1)$

Let us use the method of separation of variables. Here $y = y(x,t)$. Let us take $y = X(x)T(t)$

As solution of (1). Then

$$\frac{\partial y}{\partial x} = X^1(x)T(t); \frac{\partial^2 y}{\partial x^2} = X^{11}(x)T(t);$$

$$\frac{\partial y}{\partial t} = X(x)T^1(t); \frac{\partial^2 y}{\partial t^2} = X(x)T^{11}(t)$$

Using these in (1) we get

$$X^{11}(x)T(t) = \frac{1}{c^2}X(x)T^{11}(t)$$

$$\therefore \frac{X^{11}(x)}{X(x)} = \frac{1}{c^2}\frac{T^{11}(t)}{T(t)}$$

Since the left hand side is function of x and right hand side is a function of t the equality is possible if and only if each side is equal to the same constant (say) λ.

Hence we shall take

$$\frac{X^{11}(x)}{X(x)} = \frac{1}{c^2}\frac{T^{11}(t)}{T(t)} = \lambda$$

Let us take λ to be real. Then three cases are possible $\lambda > 0, \lambda = 0\ or\ \lambda < 0$

<u>Case 1:-</u> let $\lambda > 0$, then $\lambda = p^2 (p > 0)$

Then $$\frac{X^{11}(x)}{X(x)} = \frac{1}{c^2}\frac{T^{11}(t)}{T(t)} = p^2$$

Hence $X^{11}(x) = p^2 X(x)(i.e.,) X^{11}(x) - p^2 X(x) = 0$

$$i.e., \frac{d^2X}{dx^2} - p^2 X = 0 \Rightarrow X(x) = A_1 e^{px} + B_1 e^{-px}$$

$$Also\ T^{11}(t) - p^2 c^2 T(t) = 0$$

$$\Rightarrow T(t) = C_1 e^{pct} + D_1 e^{-pct}$$

Hence in this case, a typical solution is like

$$y(x,t) = \left(A_1 e^{px} + B_1 e^{-px}\right)\left(C_1 e^{pct} + D_1 e^{-pct}\right) ------(S.1)$$

Where A_1, B_1, C_1, D_1 are arbitary constants

Case 2:- let $\lambda - 0$ then

$$\frac{X^{11}(x)}{X(x)} = \frac{T^{11}(t)}{C^2 T(t)} = 0$$

$$\therefore X^{11}(x) = 0 \Rightarrow X(x) = A_2 + B_2 x$$

$$T^{11}(t) = 0 \Rightarrow T(t) = C_2 + D_2 t$$

$$\therefore y(x,t) = (A_2 + B_2 x)(C_2 + D_2 t) + ------(S.2)$$

Where A_2, B_2, C_2, D_2 are arbitary constants

Case 3:- Let $\lambda < 0$. Then we can write $\lambda = -p^2$ where $p > 0$ then

$$\frac{X^{11}(x)}{X(x)} = \frac{T^{11}(t)}{c^2 T(t)} = -p^2$$

$$\therefore X^{11}(x) + p^2 X(x) = 0$$

$$\Rightarrow X(x) = (A_3 \cos px + B_3 \sin px)$$

$$T^{11}(t) + p^2 c^2 T(t) = 0$$

$$\Rightarrow X(t) = (C_3 \cos pct + D_3 \sin pct)$$

Hence a typical solution in this case is

$$y(x,t) = (A_3 \cos px + B_3 \sin px)(C_3 \cos pct + D_3 \sin pct)$$

Thus the possible solution forms of equation (1) are

$$y(x,t) = \left(A_1 e^{px} + B_1 e^{-px}\right)\left(C_1 e^{pct} + D_1 e^{-pct}\right) ----(S.1)$$

$$y(x,t) = (A_2 + B_2 x)(C_2 + D_2 t) -------(S.2)$$

$$y(x,t) = (A_3 \cos px + B_3 \sin px)(C_3 \cos pct + D_3 \sin pct) ---(S.3)$$

Consider $(S.1)$ (I.e.,)

$$y(x,t) = \left(Ae^{px} + Be^{-px}\right)\left(Ce^{pct} + De^{-pct}\right)$$

Using conditions (2) $(viz)\, y(0,t) = 0\ \ for\ \ all\ \ t$

$$(A+B)\left(Ce^{pct}+De^{-pct}\right)=0 \;\; for\;\; all\;\; t$$

$$\therefore A+B=0$$

Using condition (3), $y(l,t)=0 \;\; for\;\; all\;\; t$

$$\therefore \left(Ae^{pl}+Be^{-pl}\right)\left(Ce^{pct}+De^{pct}\right)=0 \;\; for\;\; all\;\; t$$

$$\therefore Ae^{pl}+Be^{-pl}=0$$

Solving $A+B=0$

And $Ae^{pl}+Be^{-pl}=0$

We get $A=B=0$

Thus $y(x,t)=0$

This implies that there is no displacement for any x and for any t. this is impossible. Thus $(S.1)$ is not an appropriate solution

Consider $(S.2)$:

$$y(x,t)=(A+Bx)(C+Dt)$$

Using (2), $y(0,t)=0 \;\; for\;\; all\;\; t$

Hence $A(C+Dt)=0 \Rightarrow A=0$

Using (3), $y(l,t)=0 \;\; for\;\; all\;\; t$

$$\therefore (A+Bl)(C+Dt)=0 \;\; for\;\; all\;\; t$$

$$\therefore Bl(C+Dt)=0\forall t \quad \text{since A = 0}$$

Here $l \neq 0; C+Dt \neq 0 \forall t$ Hence B = 0

Thus here again $y(x,t)\equiv 0 \forall x \;\; and\;\; t$

Thus as before, this solution also is not valid

Hence $(S.2)$ is also not appropriate for the present problem

Consider $(S.3)$

$y(x,t)=(A\cos px+B\sin px)(C\cos pct+D\sin pct)$ (using condition 2)

$y(x,t)=0\forall t$

$\Rightarrow A(C\cos pct+D\sin pct)=0$

$\Rightarrow A=0$

Using condition 3

$y(l,t)=0\forall t$

$B\sin pl\left(C\cos pct + D\sin pct\right) = 0$

$if\ \ B = 0, y\left(x,t\right) = 0$ and this is invalid

Hence $\sin pl = 0$

$\therefore\ pl = n\pi$ where n = 1, 2, 3…………

Thus $p = \frac{n\pi}{l}\left(n = 1, 2, 3, ...\right)$

Thus a typical solution of (1) satisfying conditions (2) & (3) is

$$y\left(x,t\right) = \sin\frac{n\pi x}{l}\left[C_n\cos\frac{n\pi ct}{l} + D_n\sin\frac{n\pi ct}{l}\right]$$

$for\ \ n = 1, 2, 3......$

Since different solutions correspond to different positive integer n.

<u>An Important observation here :</u>

If $\left[y_n\left(x,t\right)\right]_{n=1}^{\infty}$ are functions satisfying (1) as well as conditions (2) and (3). As the equation (1) is linear. The most general solution of (1) here is $y\left(x,t\right) = \sum_{n=1}^{\infty} y_n\left(x,t\right)$

Thus the most general solution of (1) satisfying (2) & (3) is

$$y\left(x,t\right) = \sum_{n=1}^{\infty}\left[C_n\cos\frac{n\pi ct}{l} + D_n\sin\frac{n\pi ct}{l}\right]\sin\frac{n\pi x}{l} \rightarrow (6)$$

Where $C_n\ and\ D_n$ are constants to be determined using (3) and (4)

Let us use condition 4: $y\left(x,0\right) = f\left(x\right), 0 \le x \le l$

Thus putting t = 0 in (6)

$$\sum_{n=1}^{\infty} C_n\sin\frac{n\pi x}{l} = f\left(x\right), 0 \le x \le t$$

Hence $C_n = 2/l\int_0^l f\left(x\right)\sin\frac{n\pi x}{l}dx$ n = 1, 2, ……

Thus $C_n's$ are all determined

Let us consider condition (5):

$$\left(\frac{\partial y}{\partial t}\right)_{at\,t=0} = g(x)\,\forall 0 \le x \le t$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left\{\left(-C_n \sin\frac{n\pi ct}{l}\left(\frac{n\pi c}{l}\right) + D_n \cos\frac{n\pi ct}{l}\left(\frac{n\pi c}{l}\right)\right)\sin\frac{n\pi x}{l}\right\}$$

$$\left.\frac{\partial y}{\partial t}\right|_{at\,t=0} = g(x)$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(D_n \frac{n\pi c}{l}\right)\sin\frac{n\pi x}{l} = g(x), 0 \le x \le l$$

Hence $D_n = \frac{2}{n\pi c}\int_0^t g(x)\sin\frac{n\pi x}{l}\,dx \quad for\,(n = 1, 2.....)$

Thus D_n are all determined

Hence the displacement $y(x,t)$ at any point x and at any subsequent time t is given by

$$y(x,t) = \sum\left(C_n \cos\frac{n\pi ct}{l} + D_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \to (6)$$

Where $C_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \to (7)$

$$D_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx \to (8)$$

TWO DIMENSIONAL WAVE EQUATION:-

Two dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = C^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - - - - - (1)$$

Where $C^2 = T / P$, for the unknown displacement $u(x,y,t)$ of a point (x,y) of the vibrating membrane from rest $(\mu = 0)$ at time t.s

The boundary conditions (membrane fixed along the boundary in the xy- plane for all times $t \ge 0$, are $u = 0$ on the boundary ----(2)

And the initial conditions are

$$u(x,y,0) = f(x,y) : u_t(x,y,o) = g(x,y) - - - - (3)$$

$$where\ u_t = \frac{\partial u}{\partial t}$$

Now we have to find a solution of the partial differential equation (1) satisfying the conditions (2) and (3) . we shall do this in 3 steps, as follows:

Working rule to solve two – dimensional wave equation :-

Step1: By the "method of separating variables" setting $u(x,y,t)=F(x,y),G(t)$ and later $F(x,y)=H(x)Q(y)$ we obtain from (1) an ordinary differential equation for G and one partial differential equation for F, two ordinary differential equations for H & Q.

Step 2: We determine solutions of these equations that satisfy the boundary conditions (2). Step(2) to obtain a solution of (1) satisfying both (2) and (3). That is the solution of the regular membrane as follows.

The double Fourier series for $f(x,y)=\left[u(x,y,0)\right]$ is given by

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}u_{mn}(x,y,t)$$

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[B_{mn}\cos\lambda_{mn}t+B^{*}{}_{mn}\sin\lambda_{mn}t\right]\sin\frac{n\pi x}{a}\sin\frac{n\pi y}{b}$$

Hence B_{mn} and $B^{*}{}_{mn}$ are called Fourier co-efficients of $f(x,y)$ and are given by

$$B_{mn}=\frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{n\pi x}{a}\sin\frac{n\pi y}{b}dxdy, m=1,2.......;n=1,2.......$$

$$and\ B^{*}{}_{mn}=\frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy, m=1,2.......;n=1,2.......$$

1. **Find the solution of the wave equation $\frac{\partial^2 u}{\partial t^2}=C^2\frac{\partial^2 u}{\partial x^2}$ corresponding to the triangular initial deflection**

$$f(x)=\frac{2kx}{l}\ where\ 0<x<l/2$$
$$=\frac{2k}{l}(l-x)\ where\ l/2<x<l$$

and initial velocity is equal to 0.

Ans. To find $u(x,t)$ we have to solve

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}\rightarrow(1)$$

Where

$$u(0,t)=0\forall t\rightarrow(2)$$
$$u(l,t)=0\forall t\rightarrow(3)$$
$$u(x,0)=f(x)(0\le x\le l)\rightarrow(4)$$
$$\left(\frac{\partial u}{\partial t}\right)_{at\ t=0}=g(x)=0(0\le x\le l)\rightarrow(5)$$

Equation (1) can be in the form

$$u(x,t)=T(t)X(x)$$

The three solutions of (1) are

$$u(x,t)=\left(A_1e^{px}+B_1e^{-px}\right)\left(C_1e^{pct}+D_1e^{-pct}\right)----(S.1)$$
$$u(x,t)=\left(A_2+B_2x\right)\left(C_2+D_2t\right)----(S.2)$$
$$u(x,t)=\left(A_3\cos px+B_3\sin px\right)\left(C_3\cos pct+D_1\sin pct\right)----(S.3)$$

The appropriate solution is S.3

Hence $u(x,t)=\left(A\cos px+B\sin px\right)\left(C\cos pct+D\sin pct\right)$

Using (2) & (3)

$$A=0; P=\frac{n\pi}{l} \text{ where } n=1,2,3......$$

$\therefore$ The most general solution of (1) satisfying (2) & (3) is

$$u(x,t)=\sum_{n=1}^{\infty}\left(C_n\cos\frac{n\pi ct}{l}+D_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}\rightarrow(6)$$

Using (4)

$$u(x,0)=f(x)$$

$$\therefore f(x)=\sum C_n\sin\frac{n\pi x}{l}\,\forall x\in[0,l]\rightarrow(7)$$

Now we can expand the given function $f(x)$ in a half range fourier sine series for $0<x<l$

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l} \text{ where } b_n=\frac{2}{l}\int_0^l f(x).\sin\frac{n\pi x}{l}dx\rightarrow(8)$$

Comparing (7) & (8) we get $c_n=b_n$

$$\therefore c_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\left[\int_0^{l/2}\frac{2k}{l}x\sin\frac{n\pi x}{l}dx+\int_{l/2}^{l}\frac{2k}{l}(l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$=\frac{4k}{l^2}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-1\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^{l/2}+\left\{(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(-1)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right)^l_{l/2}\right\}\right]$$

$$=\frac{4k}{l^2}\left[l/2.\frac{1}{n\pi}-\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

The required solution of (1) is of the form

$$u(x,t)=\left(c_1\cos px+c_2\sin px\right)+\left(c_3\cos pat+c_4\sin pat\right)\rightarrow(6)$$

Using (2) & (3), we have

$$c_1=0 \text{ and } p=\frac{n\pi}{l} \text{ where } n=1,2,3.....$$

$\therefore$ General solution of (1) satisfying (2) & (3) is

$$u(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi at}{l} + c_4 \sin\frac{n\pi at}{l}\right) \to (7)$$

Now using condition (4) $u(x,0)=0$ we get

$$u(x,0) = 0 = c_2 \sin\frac{n\pi x}{l}(c_3 + 0)$$

$$\Rightarrow c_2 c_3 \sin\frac{n\pi x}{l} = 0 \Rightarrow c_3 = 0 \because (c_2 \neq 0) \to (8)$$

$$\textit{from } (7) \& (8)$$

$$u(x,t) = c_2 \sin\frac{n\pi x}{l}\left(0 + c_4 \sin\frac{n\pi at}{l}\right)$$

$$= c_4 \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} \quad where\, c_n = c_2 c_4$$

The most general solution of (1) is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} ----(9)$$

$$\frac{\partial(u(x,t))}{\partial t} = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right)$$

$$\frac{\partial}{\partial t} l(x,0) \sum_{n=1}^{\infty} C_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l}$$

From (5) & above result

$$\sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l}$$

$$\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l} = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l}$$

$$= \left[c_1 \frac{\pi a}{l} \sin\frac{\pi x}{l} + c_2 \frac{2\pi a}{l} \sin\frac{2\pi x}{l} + -----\right] - \left[l/2 \frac{l}{n\pi}\left(-\cos\frac{n\pi}{2}\right) - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4k}{l^2} 2 \frac{l^2}{n^2\pi^2} \sin\frac{n\pi}{2}$$

$$= \frac{8k}{n^2\pi^2} \sin\frac{n\pi}{2}$$

If $n = 2m$ (an even number) $c_{2m} = 0$

If $n = 2m+1 (an\ odd\ number), c_{2m+1} = \dfrac{8k}{(2m+1)^2 \pi^2}(-1)^m$

Thus all c_n's are determined

Using

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \ for\ 0 \le x \le l$$

$$D_n = \frac{2}{n\pi c}\int_0^l g(x) \sin\frac{n\pi x}{l} dx$$

$$= 0 \qquad \text{Since } g(x) = 0$$

Hence, $u(x,t)=\frac{8k}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\sin\frac{(m+1)\pi ct}{l}\sin\frac{(2m+1)\pi x}{l}$

1. **Solve the boundary value problem**

$u_{tt}=a^2u_{xx};0<x<l;t>0\ with\ u(0,t)=0,u(l,t)=0\ \&\ u(x,0)=0,u_t(x,0)=\sin^3\left(\frac{\pi x}{l}\right)$

Ans. $u(x,t)$ is the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}\to(1)$$

Given conditions are

$$u(0,t)=0\forall t\to(2)\ and\ \mu_t(x,0)=\sin^3\frac{\pi x}{l}\forall x\in[0,l]\to(5)$$

$$u(l,t)=0\forall t\to(3)$$

$$u(x,0)\forall 0\le x\le l\to(4)$$

Comparing the coefficients of like terms,

$$c_1\frac{\pi a}{l}=\frac{3}{4},c_2=0,c_3\left(\frac{3\pi a}{l}\right)=\frac{-1}{4},c_4,c_5---c_n=0$$

$$\Rightarrow c_1=\frac{3l}{4\pi a},c_2=0,c_3=\frac{-1}{1/2\pi a},c_4=0$$

Hence, satisfying the values in (9)

$$u(x,t)=\frac{3l}{4\pi a}\sin\frac{\pi x}{l}\sin\frac{\pi at}{l}-\frac{1}{12\pi a}\sin\frac{3\pi x}{l}\sin\frac{3\pi at}{l}$$

2. **If a string of length l is initially at rest in equillibrium position and each of its points is given the velocity $V_o\sin^3\frac{\pi x}{l}$, find the displacement $y(x,t)$**

Ans. with the explained notation, the displacement $y(x,t)$ is given by

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}\to(1)$$

$$y(0,t)=0\forall t\to(2)$$

$$y(l,t)=0\forall t\to(3)$$

$$y(x,0)=0\le x\le l\to(4)$$

$$\left.\frac{\partial y}{\partial t}\right|_{at\,t=0}=V_0\sin^3\frac{\pi x}{l}\to(5)$$

The most general solution of (1) satisfying (2) & (3) is

$$y(x,t)=\sum_{n=1}^{\infty}\left(C_n\cos\frac{n\pi ct}{l}+D_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}\to(6)$$

Using (4) we get $\sum C_n\sin\frac{n\pi x}{l}=0\forall x\in[0,l]$ which implies $C_n=0$ for all n

Now, using (5), we get

$$\sum D_n\frac{n\pi c}{l}\sin\frac{n\pi x}{l}=V_0\sin^3\frac{\pi x}{l}$$

$$=V_0\left[\frac{3}{4}\sin\frac{\pi x}{l}-\frac{1}{4}\sin\frac{3\pi x}{l}\right]$$

Hence $D_1 = \frac{-3l}{4\pi c V_o}, D_3 = \frac{-lV_0}{12\pi c}$

Hence $y(x,t) = \frac{-3lV_o}{4\pi c}\sin\frac{\pi ct}{L}\sin\frac{\pi x}{L} - \frac{lV_0}{12\pi c}\sin\frac{3\pi ct}{l}\sin\frac{\pi x}{l}$

SHORT TYPE QUESTION AND ANSWERS

Problem 1 : Write the standard forms of non – linear PDE.

Solution:

❑ **Standard Form 1**

Equations of the form $F(p,q) = 0$, not involving x, y and z are said to be in the standard form 1

❑ **Standard form 2**

i.e, $F(p,q,z) = 0$

❑ **Standard form 3:**

Equations of the form $z = px + qy + f(p,q)$

❑ **Standard Form 4 :**

Equations of the form $f_1(x,p) = f_2(y,q)$

2. **Problem 2 :** Write the three possible solutions for a one – dimensional heat equation**.**

Solution:

Let the heat equation is

$$\frac{\partial u}{\partial t} = C^2\frac{\partial^2 u}{\partial x^2} \quad \rightarrow (1)$$

By the method of separation of variables , let u(x , t) =X(x)T(t) ,then we get two ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \rightarrow (4) \text{ and } \frac{dT}{dt} - kC^2T = 0$$

Solving the above equations we get three possible solutions based on the value of k

1) When 'k' is positive and k = p^2 (say)

$$X = C_1e^{px} + C_2e^{-px}, T = C_3e^{c^2p^2t}$$

2)When 'k' is negative and k = $-p^2$ (say)

$$X = C_4\cos px + C_3\sin px, T = C_6\ e^{-c^2p^2t}$$

3) When 'k' is zero

$$X = C_7x + C_8, T = C_9$$

Problem 3 : Write the three possible solutions for a two– dimensional heat equation**.**

Solution: The two– dimensional heat equation is given by

$$\frac{\partial u}{\partial t} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \text{ where } c^2 = \frac{k}{\rho s} \text{ -}$$ where thermal conductivityρ (cal / cm sec)

$\alpha(cm)$, density ρ $\left(\frac{gr}{cm^3}\right)$, $specific\ heat\ s\ (\frac{cal}{ar}deg)$

By the method of separation of variables , let u(x , t) =X(x)T(t) ,then we get two ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \text{ and } \frac{d^2Y}{dy^2} + kY = 0$$

Solving the above equations we get three possible solutions based on the value of k

I. When k is positive and $k = p^2$, say, then,

$$X = c_1 e^{px} + c_2 e^{-px}, \qquad Y = c_3 \cos py + c_4 \sin py$$

II. When k is negative and $k = -p^2$, say, then,

$$X = c_1 \cos px + c_2 \sin px \;, Y = c_3 e^{py} + c_4 e^{-py}$$

III. When $k = 0$, we have,

$$X = c_1 x + c_2 \;, \qquad Y = c_3 y + c_4$$

Thus the various possible solutions of Laplace's equation (1) are:

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

$$u = (c_1 \cos px + c_2 \sin px)\,(c_3 e^{py} + c_4 e^{-py})$$

$$u = (c_1 x + c_2)(c_3 y + c_4)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem and the given boundary conditions.

Problem 4 : Write the three possible solutions for a two– dimensional wave equation**.**

Solution: The equation for the vibrations of a tightly stretched membrane with T, tension per unit length, m , Mass of the membrane for unit area is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \text{ where } c^2 = \frac{T}{m}$$

By the method of separation of variables , let u(x , t) =X(x)Y(y)T(t) ,then we get three ordinary differential equations

$$\frac{d^2X}{dx^2} + k^2 X = 0 \,, \; \frac{d^2Y}{dy^2} + l^2 Y = 0 \text{ and}$$

$$\frac{d^2T}{dt^2} + (k^2 + l^2)c^2 T = 0$$

The solution of these equations are respectively,

$$X = c_1 \cos kx + c_2 \sin kx$$

$$Y = c_3 \cos ly + c_4 \sin ly$$

$$T = c_5 \cos\sqrt{k^2 + l^2}\, ct + c_6 \sin\sqrt{k^2 + l^2}\, ct$$

Problem 5: Explain the method separation of variables.

- Let the given partial differential equation be $\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} \rightarrow (1)$
- Take the solution of given partial differential equation as $U(x,t) = X(x)T(t) \rightarrow (2)$ where x and t are two independent variables.
- Differentiate (2) partially with respect to 'x' and 't'

i.e., $\frac{\partial u}{\partial x} = X'T \rightarrow (3)$ and $\frac{\partial u}{\partial t} = XT' \rightarrow (4)$

- Substitute (3) and (4) in (1) we get

$X(x)\,T' = k.\,X''T \rightarrow (5)$

- Separate the variables x and t from (5) i.e., $\frac{X''(x)}{X(x)} = k.\,\frac{T'}{T} \rightarrow (6)$
- Equate L.H.S and R.H.S of (6) to some constant and then integrate them separately.
- Substitute the resultant from the above step in equation (2) to get complete solution of the given partial differential equation

Problem 6 : What are the conditions assumed in deriving one dimensional wave equation?

Solution:

i. The motion takes place entirely in one plane.
ii. We consider only transverse vibrations, the horizontal displacement of the particles of the string is negligible.
iii. The tension T is constant at all times and at all points of the deflected string.
iv. Gravitational force is negligible.
v. The effect of friction is negligible.
vi. The string is perfectly flexible.

The slope of the deflection curve at all points and at all instants is so small that $\sin\alpha$ can be replaced by α, where α the inclination of the tangents to the deflection curve

Problem 7 : State the suitable solution of the one dimensional heat equation.

Solution:

Let the heat equation is

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2} \rightarrow (1)$$

and the suitable solution is $u(x,t) = (A\cos px + B\sin px)\,e^{-a^{2}p^{2}t}$

Problem 8: A string is stretched and fastened to two points l distance apart. Motion is started by displacing the string into the form $y = y_0 \sin(\pi x/l)$ from which it is released at time $t = 0$. Formulate this problem as a boundary value problem.

Solution : The one dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are i.

$y(0, t) = 0$

ii. $y(l, t) = 0$

iii. $\frac{\partial y}{\partial t}(x,0) = 0$

iv. $y(x,0) = y_0 \sin(\pi x/l)$

Problem 9: A rod of length 20 cm whose one end is kept at 30°C and the other end is kept at 70°C is maintained so until steady state prevails. Find the steady state temperature.

Solution : In the steady state temperature the temperature will be a function of x alone

$$\therefore \frac{\partial^2 u}{\partial x^2} = 0$$

$u(x) = ax + b$

when $x = 0$, $u(0) = 30$

when $x = 20$, $u(20) = 70$

$u(x) = ax + b$

$u(0) = a0 + b$

$30 = b$

$u(20) = a20 + 30$

$70 = 20a + 30$

$20a = 40$

$a = 2$

$\therefore u(x) = 2a + 30$

Problem 10: State two dimensional Laplace equation.

Solution : The two dimensional Laplace equation is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Problem 11: What are the assumptions made before deriving the one dimensional heat equation?

Solution :Heat flows from a higher to lower temperature.

(i) The amount of heat required to produce a given temperature change in a body is proportional to the mass of the body and to the temperature change.

(ii) The rate at which heat flows through an area is proportional to the area and to the temperature gradient normal to the area.